Part V

第五篇

Supergravity

超引力

Sylvester James Gates and Konstantinos Koutrolikos

西尔维斯特·詹姆斯·盖茨与康斯坦丁诺斯·库特罗利科斯

Contents

目录

G. Dall'Agata (☑)

G. 达拉加塔 (☑)

Dip. di Fisica e Astronomia "Galileo Galilei", Università di Padova and INFN, Padova, Italy e-mail: gianguido.dallagata@pd.infn.it

意大利帕多瓦大学伽利略伽利略物理与天文系，意大利国家核物理研究所帕多瓦分部，帕多瓦，意大利电子邮箱:gianguido.dallagata@pd.infn.it

M. Zagermann

M. 扎格曼

Hamburg, Germany

德国汉堡

e-mail: Marco.Zagermann@desy.de

电子邮箱:Marco.Zagermann@desy.de

We present a short overview of the structure and couplings of $4D, \mathcal{N} = 1$ supergravity theories at the component level. We do so with as little technical machinery as possible, working directly with the physical on-shell fields and using explicit computations and geometrical reasoning to arrive at the result, highlighting the new properties of supersymmetry in the context of a gravitational theory.

本文将在分量层面简要概述 $4D, \mathcal{N}=1$ 超引力理论的结构与耦合。我们尽可能简化了技术环节，直接围绕物理在壳场展开研究，通过显式计算与几何推导得出结果，并着重阐述引力理论框架下超对称的新性质。

Keywords

关键词

Supergravity - Supersymmetry

超引力 - 超对称

Introduction

引言

Supergravity is soon going to turn 50 [2,3]. During this half of a century, it lived several lives, and it has been used and studied from various different vantage points, including the analysis of quantum gravity and black hole physics, string theory, particle physics phenomenology, cosmology, and mathematics. Each of these different approaches advanced our understanding of the features and structure of supergravity theories, and, in turn, supergravity brought new ideas and fertilized each of these fields of study.

超引力即将迎来诞生 50 周年 [2,3]。半个世纪以来，超引力历经多次发展，人们从多个不同视角对其展开研究与应用，涵盖量子引力分析、黑洞物理、弦论、粒子物理唯象学、宇宙学以及数学领域。这些不同方向的研究都推进了我们对超引力理论特征与结构的理解，反过来，超引力也为这些研究领域带来了新思想，推动了各领域的发展。

One of the main problems of the uninitiated who is interested in supergravity is that it is a rather technical subject and most of the introductory books and reviews deal with it by first emphasizing some specific mathematical formalism (like superspace and superfields or the group manifold approach or the superconformal approach) and only after some significant effort by the reader, they enter into the discussion of the physical properties of the theory. While each of these approaches has its advantages and can be at some point necessary to obtain significant progress, we felt that a more simple hands-on introduction, where every aspect is dealt with directly at the component level, emphasizing the physical features and their mathematical origin, was missing. For this reason, while there are already several great reviews that use one of the aforementioned techniques (see, for instance, [4-13]), we worked on a new physics-first introduction to the subject, which took shape in the lecture notes [1].

对于想要了解超引力的入门者而言，一个核心问题在于: 超引力是一门专业性极强的学科，现有多数入门书籍和综述在讲解时，都会先着重介绍特定的数学形式体系 (比如超空间与超场、群流形方法或者超共形方法)，需要读者投入大量精力之后，才会开始讨论该理论的物理性质。尽管这些方法各有优势，在某些情况下对取得重大进展而言是必要的，但我们发现，目前仍缺少一种更简洁、更实操的入门讲解——这种讲解直接在分量层面处理各个方面，着重突出物理性质及其数学起源。因此，尽管目前已经有多部出色的综述采用了上述某一种方法 (例如参见 [4-13])，我们还是着手编写了一套以物理为先的超引力新入门讲义，最终成型为讲义 [1]。

The current chapter is a short redacted excerpt of the more detailed and complete presentation given in [1]. Here, we mainly focus on the very basic ingredients that are needed for a first introduction to the subject, which we hope will work as an invitation for the reader to deepen their knowledge of the subject.

本章是 [1] 中更详尽完整内容的精简整理摘录。我们在此主要聚焦超引力入门所需的最基础内容，希望能吸引读者进一步深入研究超引力。

We also stress that in this collection, the reader is also going to find short introductions to some of the alternative approaches mentioned above, as well as various applications.

我们还要强调，本文集中，读者也可以找到上述部分替代方法的简介，以及各类相关应用。

What Is Supergravity?

什么是超引力?

Depending on which aspect one wants to emphasize, one could define supergravity theories in three different ways:

根据想要强调的不同方面，可以通过三种不同方式定义超引力理论:

1. Supergravity theories are supersymmetric field theories with gravity where the supersymmetry transformations act non-trivially also on the gravitational field.

1. 超引力理论是包含引力的超对称场论，超对称变换也会对引力场产生非平凡作用。

2. Supergravity theories are supersymmetric field theories in which supersymmetry is realized not only as a global (rigid) symmetry but as a local (gauge) symmetry.

2. 超引力理论是超对称场论，其中超对称不仅作为整体 (刚性) 对称性实现，还作为局部 (规范) 对称性实现。

3. Assuming finitely many fields and couplings as well as consistency with unitarity and diverse spacetime backgrounds, supergravity theories are the only known theories with consistently interacting spin-3/2-fields. In supergravity, these spin- 3/2 fields are called gravitino fields (or gravitini). For the sake of readability, we do not distinguish carefully here between spin and helicity, i.e., ”spin s ” should be understood as ”helicity $\pm s$ ” in the massless case.

3. 在假设场和耦合数量有限、满足幺正性一致性，且适用于多种时空背景的条件下，超引力是目前已知唯一能够让自旋 3/2 场自洽相互作用的理论。在超引力中，这些自旋 3/2 场被称为引力微子场 (或 gravitini)。为方便阅读，我们在此不严格区分自旋和螺旋度，也就是说，在无质量情形下，“自旋 s ”应理解为“螺旋度 $\pm s$ ”。

In this section, we illustrate why these three apparently different characterizations describe essentially the same class of theories. We do so in the assumption that gravity is described by Einstein's general theory of relativity, so that supergravity actions consist of the Einstein-Hilbert term plus a restricted class of matter actions coupled to gravity. To this end, we revisit the simplest globally supersymmetric field theory in four dimensions, the free massless Wess-Zumino model for one chiral multiplet, and discuss how this theory has to be changed when supersymmetry is turned into a local symmetry, following Definition 2. As we will see, making supersymmetry local by a simple iterative procedure (the "Noether method") directly exhibits the need for the gravitino field (cf. Definition 3) and its superpartner, the graviton (cf. Definition 1), and suggests the supersymmetry transformation laws of these fields. We end this section with a discussion of some basic properties of the gravitino field.

本节我们将说明，这三种看似不同的刻画本质上描述的是同一类理论。我们的讨论基于引力由爱因斯坦广义相对论描述的假设，因此超引力作用量由爱因斯坦-希尔伯特项加上耦合到引力的一类限定物质作用量构成。为此，我们重新考察四维最简单的整体超对称场论，即单个手征多重态的自由无质量韦斯-祖米诺模型，然后遵循定义 2 的思路，讨论当超对称变为局部对称性时，该理论需要如何修改。我们将会看到，通过简单的迭代步骤 (“诺特定方法”) 将超对称局部化，会直接表明需要引入引力微子场 (参见定义 3) 及其超伙伴引力子 (参见定义 1)，并给出这些场的超对称变换规律。最后我们在本节末尾讨论引力微子场的一些基本性质。

Promoting Supersymmetry to a Local Symmetry

将超对称性推广为定域对称性

Throughout this chapter, we use anti-commuting four-component Majorana spinors to describe fermionic degrees of freedom. Our conventions are summarized in Appendix "Appendix: Conventions, Spinors, and Useful Relations" and follow the textbook [1], where many further details can be found.

本章通篇使用反对易的四分量马约拉纳旋量描述费米子自由度。我们的约定总结在附录「附录: 约定、旋量与有用关系式」中，遵循教科书 [1] 的设定，更多细节可参阅该教材。

Consider the free massless Wess-Zumino model for one chiral multiplet (ϕ, χ) , where $\phi(x)$ is a complex scalar and $\chi(x)$ a Majorana spinor field, with Lagrangian.

考虑单个手征多重态 (ϕ, χ) 的无质量自由韦斯-祖米诺模型，其中 $\phi(x)$ 是复标量，$\chi(x)$ 是马约拉纳旋量场，拉格朗日量为。

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi^* - \left(\overline{\chi}_R \partial\!\!\!/ \chi_L + \overline{\chi}_L \partial\!\!\!/ \chi_R\right). \tag{1}$$

We recall that the mass dimensions of these fields are $D[\phi] = 1$ and $D[\chi] = 3/2$.

我们回顾这些场的质量量纲为 $D[\phi] = 1$ 和 $D[\chi] = 3/2$ 。

The Lagrangian (1) is invariant up to a total derivative under the supersymmetry variations

拉格朗日量 (1) 在超对称变换下，只差一个全微分是不变的

$$\delta_\varepsilon\phi = \bar{\varepsilon}_L\chi_L \;\Leftrightarrow\; \delta_\varepsilon\phi^* = \bar{\varepsilon}_R\chi_R \tag{2}$$

$$\delta_\varepsilon\chi_L = \frac{1}{2}\delta\phi\varepsilon_R \;\Leftrightarrow\; \delta_\varepsilon\chi_R = \frac{1}{2}\delta\phi^*\varepsilon_L. \tag{3}$$

We remark that, since we are not using auxiliary fields, the supersymmetry algebra closes only on-shell:

注意，由于我们没有使用辅助场，超对称代数仅在壳闭合:

$$\left[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}\right]\phi = \frac{1}{2}\left(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\right)\partial_\mu\phi$$

$$\left[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}\right]\chi_L = \frac{1}{2}\left(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\right)\partial_\mu\chi_L + [\ldots]\,\partial'\chi_L,$$

where [...] denotes a non-vanishing expression of the fields and supersymmetry parameters. The last term then vanishes due to the field equation $\partial\chi_L = 0$, and one obtains the usual susy algebra $\left[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}\right] = \frac{1}{2}\left(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\right)\partial_\mu$ on all fields. Note that $D[\varepsilon] = -1/2$. In our conventions,(3) is equivalent to

其中 [...] 是场与超对称参数的非零表达式。最后一项因场方程 $\partial\chi_L = 0$ 消失，我们得到所有场上的通常超对称代数 $\left[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}\right] = \frac{1}{2}\left(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\right)\partial_\mu$ 。注意 $D[\varepsilon] = -1/2$ 。在我们的约定下，(3) 等价于

$$\delta_\varepsilon\overline{\chi}_L = -\frac{1}{2}\bar{\varepsilon}_R\partial\phi \;\Leftrightarrow\; \delta_\varepsilon\overline{\chi}_R = -\frac{1}{2}\bar{\varepsilon}_L\partial\phi^*. \tag{4}$$

To check the invariance of the Lagrangian explicitly we write the fermionic term of the Lagrangian (1) as $\mathcal{L}_{\text{fer}} = -\overline{\chi}_R\partial\!\!\!/\chi_L + \partial_\mu\left(\overline{\chi}_R\right)\gamma^\mu\chi_L$ and trace the terms involving ε_L , which come only from the variation of ϕ and $\overline{\chi}_R$, because those proportional to ε_R follow by Hermitian conjugation:

为了显式验证拉格朗日量的不变性，我们将拉格朗日量 (1) 的费米子项写为 $\mathcal{L}_{\text{fer}} = -\overline{\chi}_R\partial\!\!\!/\chi_L + \partial_\mu\left(\overline{\chi}_R\right)\gamma^\mu\chi_L$ ，追踪所有包含 ε_L 的项，这类项仅来自 ϕ 和 $\overline{\chi}_R$ 的变分，因为正比于 ε_R 的项可通过厄米共轭得到:

$$\delta\mathcal{L} = -\partial_\mu(\delta\phi)\,\partial^\mu\phi^* - \delta\overline{\chi}_R\delta\chi_L + \partial_\mu\left(\delta\overline{\chi}_R\right)\gamma^\mu\chi_L + \text{ h.c.} \tag{5}$$

Integrating by parts the first and the second terms gives, using (2), (3), and (186),

对第一项和第二项做分部积分，利用 (2)、(3) 和 (186) 可得

$$\delta\mathcal{L} = \delta\phi\Box\phi^* + 2\partial_\mu\left(\delta\overline{\chi}_R\right)\gamma^\mu\chi_L + \partial_\mu\underbrace{\left(-\delta\phi\partial^\mu\phi^* - \delta\overline{\chi}_R\gamma^\mu\chi_L\right)}_{\equiv\mathcal{K}^\mu} + \text{ h.c.}$$

$$= -\partial_\mu\left(\overline{\varepsilon}_L\right)\partial\!\!\!/\phi^*\gamma^\mu\chi_L + \partial_\mu\mathcal{K}^\mu + \text{ h.c.} \tag{6}$$

As promised, the result is that under global supersymmetry, where the supersymme-try parameter is constant, $\partial_\mu\varepsilon = 0$, the Lagrangian transforms into a total derivative:

正如预期，结果是: 在整体超对称 (超对称参数为常数) 即 $\partial_\mu\varepsilon = 0$ 下，拉格朗日量变换后只差一个全微分:

$$\delta_\varepsilon\mathcal{L} = \partial_\mu\left(\mathcal{K}^\mu + \mathcal{K}^{\mu*}\right) \equiv \partial_\mu K^\mu. \tag{7}$$

When dealing with local supersymmetry, however, the parameter ε becomes a local function of the coordinates, $\varepsilon = \varepsilon(x)$, and the Lagrangian is no longer invariant up to a total derivative. The new non-invariant part of the Lagrangian is

但处理定域超对称时，参数 ε 变为坐标的定域函数 $\varepsilon = \varepsilon(x)$ ，拉格朗日量不再只差一个全微分保持不变。拉格朗日量新增的非不变部分为

$$\delta_\varepsilon\mathcal{L}_{\text{ new }} = \left(\partial_\mu\overline{\varepsilon}\right)j^\mu = \left(\partial_\mu\overline{\varepsilon}_L\right)j_L^\mu + \left(\partial_\mu\overline{\varepsilon}_R\right)j_R^\mu, \tag{8}$$

where

其中

$$j_L^\mu \equiv -\delta\phi^*\gamma^\mu\chi_L,\ j_R^\mu \equiv -\delta\phi\gamma^\mu\chi_R \tag{9}$$

give the super-Noether current $j^\mu = j_L^\mu + j_R^\mu$. In fact, it can be easily checked that this supercurrent is a conserved current, namely, that $\partial_\mu j^\mu = 0$, upon using the equations of motion for the fields ϕ and χ . It should also be noted that the dimension of these currents is $D\left[j_{L,R}^\mu\right] = 7/2$.

给出超诺特流 $j^\mu = j_L^\mu + j_R^\mu$ 。实际上不难验证，利用场 ϕ 和 χ 的运动方程，该超流是守恒流，即满足 $\partial_\mu j^\mu = 0$ 。还需要注意，这些流的量纲是 $D\left[j_{L,R}^\mu\right] = 7/2$ 。

We can now apply Noether's method and associate to the supercurrent (9) a gauge field that compensates the non-invariance of the Lagrangian (8). This gauge field, $\psi_{\mu\alpha}$, has to have a spinorial index (i.e., the index $\alpha = 1,2,3,4$, which we will suppress again in the following) and a spacetime index $(\mu = 0,1,2,3)$, such that

我们现在可以应用诺特定理，将一个规范场与超流 (9) 关联起来，这个规范场会抵消拉格朗日量 (8) 的非不变性。该规范场 $\psi_{\mu\alpha}$ 必须带有一个旋量指标 (即指标 $\alpha = 1,2,3,4$ ，我们在后续讨论中会再次省略该指标) 和一个时空指标 $(\mu = 0,1,2,3)$ ，因此

$$\delta_\varepsilon \psi_{\mu L,R} = M_P \partial_\mu \varepsilon_{L,R},\ \delta_\varepsilon \overline{\psi}_{\mu L,R} = M_P \partial_\mu \bar{\varepsilon}_{L,R}, \tag{10}$$

where M_P is a mass parameter that is needed to relate the mass dimension 3/2 of the fermionic field ψ_μ and the dimension of the supersymmetry parameter $D[\varepsilon] = -1/2$. As suggested by the notation, M_P will later be identified with the (reduced) Planck mass.

其中 M_P 是一个质量参数，用来关联费米场 ψ_μ 的质量量纲 3/2 和超对称参数 $D[\varepsilon] = -1/2$ 的量纲。正如记号所暗示的，M_P 后续会被对应为 (约化) 普朗克质量。

The Noether procedure tells us that we need to add a new piece to the Lagrangian:

诺特过程告诉我们，需要在拉格朗日量中新增一项:

$$\mathcal{L}'_{\mathrm{WZ}} = -\frac{1}{M_P}\left(\overline{\psi}_{\mu L} j^\mu_L + \overline{\psi}_{\mu R} j^\mu_R\right). \tag{11}$$

Again M_P is needed to get a Lagrangian density whose total mass dimension is 4, and this dimensionful coupling in the action can be viewed as a first sign that we eventually need gravity in local supersymmetry.

同样，M_P 是用来得到总质量量纲为 4 的拉格朗日密度，作用量中的这个有量纲耦合可以看作一个初步信号，说明局域超对称最终必然包含引力。

Using (10) in the variation of (11), we now precisely compensate the variation of the original Wess-Zumino multiplet, but now there is a new piece to compensate in the variation of (11) from $\delta_\varepsilon j^\mu_{R,L}$, which is in general non-vanishing. To see this, it suffices to consider the variation of the term $\overline{\psi}_{\mu L} j^\mu_L$ that is quadratic in the scalar fields. This term comes from the variation of χ_L inside j^μ_L :

将 (10) 代入 (11) 的变分，我们恰好可以抵消原韦斯-祖米诺多重态的变分，但现在 $\delta_\varepsilon j^\mu_{R,L}$ 带来的 (11) 变分中出现了需要抵消的新项，该项一般不为零。想要看清这一点，只需考虑标量场二次项 $\overline{\psi}_{\mu L} j^\mu_L$ 的变分，这个项来自 j^μ_L 中 χ_L 的变分:

$$\overline{\psi}_{\mu L} \Im \phi^* \gamma^\mu \delta_\varepsilon \chi_L = \frac{1}{2} \overline{\psi}_{\mu L} \gamma^\nu \gamma^\mu \gamma^\rho \varepsilon_R \partial_\nu \phi^* \partial_\rho \phi = \overline{\psi}_{\mu L} \gamma_\nu \varepsilon_R T^{\mu\nu} + \dots \tag{12}$$

where, using some gamma matrix algebra,

其中利用了伽马矩阵代数，可得

$$T^{\mu\upsilon} = \partial^{(\mu} \phi \partial^{\upsilon)} \phi^* - \frac{1}{2} \eta^{\mu\upsilon} \left(\partial_\sigma \phi \partial^\sigma \phi^*\right) \tag{13}$$

and the dots stand for terms involving $\gamma^{\upsilon\mu\rho}$. One can show that variations bilinear in χ likewise give the energy momentum tensor for the field χ . So,

省略号代表包含 $\gamma^{\upsilon\mu\rho}$ 的项。可以证明，χ 的双线性变分同理给出场 χ 的能量动量张量。因此，

$$\delta \mathcal{L}'_{\mathrm{WZ}} \sim \frac{1}{M_P} \bar{\varepsilon} \gamma_\mu \psi_\upsilon T^{\mu\upsilon} + \dots \tag{14}$$

In order to cancel this term, we now introduce a new current which is a symmetric tensor $g_{\mu\nu}$ with transformation rule

为了抵消这一项，我们现在引入一个新的流，它是对称张量 $g_{\mu\nu}$，满足变换规则

$$\delta g_{\mu\upsilon} \sim \frac{1}{M_P}\bar{\varepsilon}\gamma_{(\mu}\psi_{\upsilon)} \tag{15}$$

and add a new piece to the Lagrangian with a coupling between the tensor field $g_{\mu\nu}$ and the energy momentum tensor:

并在拉格朗日量中新增张量场 $g_{\mu\nu}$ 与能量动量张量耦合的项:

$$\mathcal{L}''_{\mathrm{WZ}} \sim -g_{\mu\nu}T^{\mu\nu} \tag{16}$$

As only the spacetime metric can couple to the energy momentum tensor, local supersymmetry requires the coupling of the Wess-Zumino multiplet to gravity described by a dynamical spacetime metric, $g_{\mu\nu}$, and ψ_μ must be its superpartner, the gravitino, as follows from the transformation law (15). As in ordinary gauge theories, one also adds kinetic terms for these new "gauge" fields, and we thus expect a final result of the form

由于只有时空度规可以与能量动量张量耦合，根据变换规则 (15)，局域超对称要求韦斯-祖米诺多重态与由动力学时空度规描述的引力耦合，$g_{\mu\nu}$，而 ψ_μ 必然是它的超伙伴——引力微子。和普通规范理论一样，我们也需要为这些新的"规范场"添加动能项，因此我们预期最终结果具有如下形式

$$\mathcal{L} = \underbrace{\mathcal{L}_{\text{kin}}(\phi) + \mathcal{L}_{\text{kin}}(\chi)}_{\mathcal{L}'_{\mathrm{WZ}}} + \underbrace{\mathcal{L}_{\text{int}}(\phi, \chi, g_{\mu\nu}, \psi_\mu)}_{\mathcal{L}'_{\mathrm{WZ}} + \mathcal{L}''_{\mathrm{WZ}}} \tag{17}$$

$$+\mathcal{L}_{\text{kin}}(g_{\mu\nu}) + \mathcal{L}_{\text{kin}}(\psi_\mu)$$

where the dots indicate possible further interaction terms.

其中省略号代表可能存在的其他相互作用项。

We used the chiral multiplet to guess the supersymmetry transformation rules of the supergravity multiplet. These rules, however, should hold also in the absence of the chiral multiplet, and we thus arrive at a motivated guess for the Lagrangian and transformation laws of pure $\mathcal{N} = 1$ supergravity:

我们借助手征多重态推导出了超引力多重态的超对称变换规则。但这些规则在不存在手征多重态时也应当成立，由此我们得到了纯 $\mathcal{N} = 1$ 超引力的拉格朗日量和变换规则的合理猜想:

$$\mathcal{L}_{\text{pure sugra}} = \underbrace{\mathcal{L}_{\text{kin}}(g_{\mu\upsilon})}_{\frac{M_P^2}{2}\sqrt{-g}R} + \underbrace{\mathcal{L}_{\text{kin}}(\psi_\mu)}_{-\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\upsilon\rho}\partial_\upsilon\psi_\rho|_{\text{cov}}} + \mathcal{L}_{\text{int}}(g_{\mu\upsilon}, \psi_\mu), \tag{18}$$

using

利用

$$\delta g_{\mu\upsilon} \simeq \frac{1}{M_P}\bar{\varepsilon}\gamma_{(\mu\psi_\upsilon)}|_{\text{cov}} \tag{19}$$

and

以及

$$\delta\psi_\mu \simeq M_P\partial_\mu\varepsilon|_{\text{cov}}, \tag{20}$$

where cov stands for a proper spacetime covariantization and $\mathcal{L}_{\text{int}}$ denotes possible interaction terms that are not contained in the covariantizations of the kinetic terms (e.g., four-Fermion terms). This spacetime covariantization is done in the vierbein formalism (cf. Appendix ”Vierbein and Cartan’s Formalism” and ”Spinors in Curved Spacetime”) and leads to the expressions (23), (26), and (36). Moreover, as we will see in section ”Minimal Supergravity in Four Dimensions,” also the additional interaction terms not related to spacetime covariantization can elegantly be absorbed into the covariantized kinetic terms by working with covariant derivatives with non-trivial torsion. Before we come to this, however, let us briefly pause and take a quick look at some basic aspects of the gravitino field.

其中 cov 表示合适的时空协变性，$\mathcal{L}_{\text{int}}$ 表示动能项协变性之外未包含的可能相互作用项 (例如四费米子项)。时空协变性是在标架形式下完成的 (参见附录“标架与嘉当形式体系”和“弯曲时空的旋量”)，最终得到表达式 (23)、(26) 和 (36)。此外，正如我们在“四维最小超引力”一节将要看到的，利用带非平凡挠率的协变导数，甚至不关联时空协变性的额外相互作用项也能优雅地被吸收进协变动能项中。不过，在讨论这一点之前，让我们先稍作停顿，快速了解引力微子场的一些基本性质。

Some Remarks on the Gravitino Field and Gravitino Multiplets

关于引力微子场与引力微子多重态的若干注记

A vector-spinor field, $\psi_{\mu\alpha}(x)$, a priori has 16 degrees of freedom. The action for a free vector-spinor field in Minkowski spacetime is the Rarita-Schwinger action [14]

一个矢量旋量场 $\psi_{\mu\alpha}(x)$ 先验地具有 16 个自由度。闵可夫斯基时空下自由矢量旋量场的作用量是拉里塔-施温格作用量 [14]

$$\mathcal{L}_{3/2} = -\frac{1}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\partial_\upsilon\psi_\rho + \frac{1}{2}m_{3/2}\overline{\psi}_\mu\gamma^{\mu\upsilon}\psi_\upsilon, \tag{21}$$

where $m_{3/2}$ is the physical mass of the corresponding particle in Minkowski spacetime. A careful analysis of the equations of motion shows that $\psi_{\mu\alpha}$ only propagates two physical degrees of freedom in the massless case, corresponding to states of helicity $\pm 3/2$, and four physical degrees of freedom in the massive case, corresponding to the polarization states of a massive spin-3/2 particle. The other degrees of freedom are either off-shell or auxiliary, as required to write a Lorentz-invariant action [15].

其中 $m_{3/2}$ 是闵可夫斯基时空下对应粒子的物理质量。对运动方程的细致分析表明，在无质量情形下 $\psi_{\mu\alpha}$ 仅传播两个物理自由度，对应螺旋度为 ±3/2 的态；在有质量情形下传播四个物理自由度，对应有质量自旋 3/2 粒子的极化态。其余自由度或是离壳的，或是辅助自由度，这是写下洛伦兹不变作用量所要求的 [15]。

In the massless case, the Rarita-Schwinger action only consists of the kinetic term and is thus invariant under the gauge symmetry

在无质量情形下，拉里塔-施温格作用量仅包含动能项，因此在以下规范对称性下保持不变

$$\delta\psi_\mu = \partial_\mu \Lambda \tag{22}$$

where $\Lambda(x)$ is an arbitrary Majorana spinor. Just as for vector gauge bosons, this gauge invariance is necessary to eliminate longitudinal polarization states and has to be preserved by interactions so as to respect unitarity. In the context of supergravity, where $\psi_{\mu\alpha}(x)$ is the gravitino, the gauge symmetry (22) is simply local supersymmetry.

其中 $\Lambda(x)$ 是任意马约拉纳旋量。和矢量规范玻色子一样，这种规范不变性对于消除纵极化态是必要的，且必须被相互作用保留，以满足幺正性要求。在超引力的框架下，当 $\psi_{\mu\alpha}(x)$ 是引力微子时，规范对称性 (22) 恰好就是局域超对称性。

While we encountered the gravitino as the superpartner of the helicity ±2 graviton, one might also wonder whether it would be possible to write down sensible field theories where the superpartner of a helicity ±3/2 particle is instead a helicity ±1 particle described by a vector field, $A_\mu(x)$. Such a multiplet is referred to as a gravitino multiplet, and a globally supersymmetric free field theory for this multiplet indeed exists. As soon as one tries to introduce interactions, however, the gauge invariance (22) must be promoted to an additional local supersymmetry, and one arrives at a theory with several local supersymmetries, i.e., extended supergravity (Consistent interacting theories including the gravitino may also be obtained by allowing for higher spin fields and couplings between an infinite number of fields [16], but this is not of interest for our discussion.).

虽然我们遇到的引力微子是螺旋度 ±2 引力子的超对称伙伴，但人们也会好奇: 是否存在合理的场论，其中螺旋度 ±3/2 粒子的超对称伙伴是由矢量场 $A_\mu(x)$ 描述的螺旋度 ±1 粒子？这样的多重态被称为引力微子多重态，而且针对该多重态的整体超对称自由场论确实存在。然而，只要人们尝试引入相互作用，规范对称性 (22) 就必须提升为额外的局域超对称性，最终会得到一个拥有多个局域超对称性的理论，也就是扩展超引力 (包含引力微子的自洽相互作用理论也可以通过允许高自旋场和无穷多场之间的耦合得到 [16]，但这不在我们的讨论范围内。)

Minimal Supergravity in Four Dimensions

四维极小超引力

Having clarified that local supersymmetry requires the coupling to gravity, we now want to show how to write down the minimal supersymmetric model, which involves only the gravity multiplet. This represents the minimal supersymmetric extension of general relativity and provides a sufficient setup to investigate and illustrate a number of general features of supergravity theories, which remain valid in the presence of additional matter multiplets, additional supersymmetries, or more general spacetime dimensions.

在明确了局域超对称性要求与引力耦合后，我们现在来说明如何写出仅包含引力多重态的极小超对称模型。这是广义相对论的极小超对称推广，它提供了足够的框架来研究和说明超引力理论的诸多普遍性质，这些性质在存在额外物质多重态、额外超对称性或更一般时空维度的情况下仍然成立。

We will also show explicitly the calculations needed to prove supersymmetry invariance as they clarify the origin and meaning of a series of structures common to all supergravity theories. Once more, additional details and extensions can be found in [1].

我们还会明确给出证明超对称性不变性所需的计算，因为这些计算阐明了所有超引力理论共有的一系列结构的起源与意义。更多补充细节和拓展内容可以参阅文献 [1]。

The Minimal Action

最小作用量

This section is reprinted from [1] ©2021 Springer-Verlag GmbH Germany, part of Springer Nature. Reproduced with permissions. All rights reserved. In the following, we will often use the language of differential forms in order to simplify calculations. For instance, the action we want to supersymmetrize must contain the Einstein-Hilbert S_{EH} and Rarita-Schwinger S_{RS} actions,

本节重印自文献 [1] ©2021 施普林格·自然旗下德国施普林格出版社，经授权转载，版权所有。下文我们将经常使用微分形式语言简化计算。例如，我们要做超对称化的作用量必须包含爱因斯坦-希尔伯特 S_{EH} 作用量和拉里塔-施温格 S_{RS} 作用量，

$$S = \int d^4x\,(\mathcal{L}_{EH} + \mathcal{L}_{RS}) = \int d^4x e\left(\frac{M_P^2}{2}R - \frac{1}{2}\overline{\psi}_\mu \gamma^{\mu\upsilon\rho} D_\upsilon \psi_\rho\right), \tag{23}$$

which, in the language of differential forms, become

用微分形式语言可写为

$$S = S_{EH} + S_{RS} = \frac{M_P^2}{4}\int R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} + \frac{i}{2}\int e^a \wedge \overline{\psi} \wedge \gamma_5 \gamma_a D\psi. \tag{24}$$

Here, the indices $a, b, c, \ldots$ are local Lorentz indices referring to orthonormal frames, $e^a = e^a_\mu dx^\mu$, with constant epsilon tensor, $\varepsilon_{0123} = 1$, and R^{ab} and ψ denote, respectively, the curvature two-form (175) of the spin connection and the gravitino one-form $\psi \equiv \psi_\mu dx^\mu . D$ denotes the Lorentz-covariant derivative (210). For further details on the formalism and our conventions, the reader is referred to the Appendix.

此处，指标 $a, b, c, \ldots$ 是对应标准正交标架的局域洛伦兹指标，$e^a = e^a_\mu dx^\mu$ 带有常 ε 张量，$\varepsilon_{0123} = 1$ ，其中 R^{ab} 和 ψ 分别代表自旋联络的曲率二形式 (175)，引力微子一形式 $\psi \equiv \psi_\mu dx^\mu . D$ 代表洛伦兹协变导数 (210)。关于形式体系和我们的约定的更多细节，读者可参考附录。

Since we are coupling the spin 3/2 field ψ_μ to gravity, the covariant derivative in its kinetic term should a priori be the full covariant derivative, ∇ , and not just the Lorentz-covariant derivative, D , we have used in the above expressions. The full covariant derivative ∇ contains both the Levi-Civita connection, Γ , coupling to the vector index μ of the gravitino, and the spin connection, ω , coupling to the (suppressed) spinor index. However, even if we had used ∇ , it would appear in the action only in antisymmetrized form,

由于我们是将自旋 3/2 场 ψ_μ 耦合到引力，其动能项中的协变导数按理说应当是全协变导数 ∇ ，而非我们在上述表达式中使用的洛伦兹协变导数 D 。全协变导数 ∇ 同时包含列维-奇维塔联络 Γ (耦合到引力微子的矢量指标 μ) 和自旋联络 ω (耦合到 (未写出的) 旋量指标)。但即便我们使用了 ∇ ，它也只会以反对称化的形式出现在作用量中,

$$\nabla_{[\upsilon}\psi_{\rho]} = \partial_{[\upsilon}\psi_{\rho]} + \frac{1}{4}\omega_{[\upsilon}^{ab}\gamma_{ab}\psi_{\rho]} - \Gamma_{[\upsilon\rho]}^{\sigma}\psi_\sigma, \tag{25}$$

and the last term is identically zero so that $\nabla_{[\upsilon}\psi_{\rho]} = D_{[\upsilon}\psi_{\rho]}$, and we can indeed use the Lorentz-covariant derivative D in the kinetic term of the gravitino. In fact, the Levi-Civita connection in terms of Christoffel symbols will never really appear in the following. It is important here that Γ really denotes the torsion-free Levi-Civita connection. As we will see later, it is useful to include a torsion piece bilinear in the gravitini in the spin connection (but not in the connection Γ , which should stay torsion-free). The connections defined by Γ and ω are then no longer equivalent connections.

且最后一项恒为零，即得 $\nabla_{[\upsilon}\psi_{\rho]} = D_{[\upsilon}\psi_{\rho]}$ ，因此我们确实可以在引力微子的动能项中使用洛伦兹协变导数 D 。事实上，以克里斯托费尔符号表示的列维-奇维塔联络在后文根本不会出现。此处需要注意 Γ 确实表示无挠列维-奇维塔联络。我们在后文会看到，在自旋联络中引入一个引力微子双线性的挠项是很有用的 (但在联络 Γ 中不需要，它应当保持无挠)。此时由 Γ 和 ω 定义的联络就不再是等价联络了。

We now discuss the invariance under supersymmetry of (23). We start by making one simple assumption that is motivated by our previous discussion on the gravitino being the gauge field of supersymmetry. This means that the gravitino transformation rule should be proportional to the (covariant) derivative of the supersymmetry parameter

我们现在讨论 (23) 在超对称下的不变性。我们先从一个简单的假设开始，这个假设来自我们之前的讨论: 引力微子是超对称的规范场。这意味着引力微子的变换规则应当正比于超对称参数的 (协变) 导数

$$\delta_\varepsilon\psi_\mu = M_P D_\mu\varepsilon \equiv M_P\left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\varepsilon, \tag{26}$$

with the conjugate field satisfying

共轭场满足

$$\delta_\varepsilon\overline{\psi}_\mu = M_P\left(\partial_\mu\bar{\varepsilon} - \frac{1}{4}\bar{\varepsilon}\gamma_{ab}\omega_\mu^{ab}\right) \equiv M_P\overline{D_\mu\varepsilon}. \tag{27}$$

Having specified only the gravitino supersymmetry transformation so far, the next thing we would like to obtain is the transformation rule of the vierbein. We could simply make an educated guess in line with our considerations leading to Eq. (15), but let us try to actually derive the vierbein transformation law from what we already have. From the variation of S_{EH}, we see that the only contribution with δe^a comes multiplied by the curvature R^{ab}. We therefore try to single out from δS_{RS} all possible terms that give the same type of contributions proportional to the curvature of the spin connection. Supersymmetry invariance will then determine δe^a, and we will then check the invariance of the full action.

到目前为止我们只确定了引力微子的超对称变换，接下来我们需要得到 vierbein(标架) 的变换规则。我们当然可以顺着得到式 (15) 的思路做一个合理猜测，但我们不妨尝试从已经得到的结论实际推导出 vierbein 的变换规律。从 S_{EH} 的变分可以看出，唯一含 δe^a 的项是乘以曲率 R^{ab} 得到的项。因此我们尝试从 δS_{RS} 中分离出所有能给出同类贡献、即正比于自旋联络曲率的项。之后超对称不变性会确定 δe^a，我们再验证全作用量的不变性。

The variation of the gravitini in the Rarita-Schwinger Lagrangian gives

拉里塔-施温格拉格朗日量中引力微子的变分给出

$$\delta\mathcal{L}_{RS} = -\frac{e}{2}\overline{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\delta\psi_\rho - \frac{e}{2}\overline{\delta\psi_\mu}\gamma^{\mu\nu\rho}D_\nu\psi_\rho + \dots$$

(28)

$$= -\frac{e}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}D_\upsilon\delta\psi_\rho - \frac{e}{2}\overline{D_\upsilon\psi_\rho}\gamma^{\mu\upsilon\rho}\delta\psi_\mu + \dots,$$

where we used the identity $\overline{\chi}\gamma^{\mu\upsilon\rho}\lambda = \overline{\lambda}\gamma^{\mu\upsilon\rho}\chi$ for anti-commuting Majorana spinors and the dots refer to the variations of the vierbein, δe_μ^a, and the spin connection, ω_μ^{ab}, which we do not consider for now because they give terms that are not of the form we need. Inserting (26) in (28), we obtain

其中我们对反对易马约拉纳旋量使用了恒等式 $\overline{\chi}\gamma^{\mu\upsilon\rho}\lambda = \overline{\lambda}\gamma^{\mu\upsilon\rho}\chi$，省略号代表 vierbein(标架) δe_μ^a 和自旋联络 ω_μ^{ab} 的变分，我们暂时不讨论这些变分，因为它们给出的项不属于我们需要的形式。将 (26) 代入 (28)，我们得到

$$\delta\mathcal{L}_{RS} = -M_P\frac{e}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}D_\upsilon D_\rho\varepsilon - M_P\frac{e}{2}\overline{D_\upsilon\psi_\rho}\gamma^{\mu\upsilon\rho}D_\mu\varepsilon + \dots \tag{29}$$

Integrating the last term by parts, we can replace it by

对最后一项分部积分后，我们可以将其替换为

$$-\partial_\mu\left(\frac{e}{2}M_P\overline{D_\upsilon\psi_\rho}\gamma^{\mu\upsilon\rho}\varepsilon\right) + \frac{e}{2}M_P\overline{D_\mu D_\upsilon\psi_\rho}\gamma^{\mu\upsilon\rho}\varepsilon \tag{30}$$

plus terms involving derivatives of the vielbein, $D_\mu e_\upsilon^a$, which we also neglect in this first step, because they will not give contributions proportional to the curvature R^{ab}. The equivalence of (30) to the last term in (29) can easily be checked by recalling either that the γ-matrices are covariantly constant in the sense that

加上包含标架导数的项 $D_\mu e_\upsilon^a$，我们在第一步同样忽略这些项，因为它们不会给出正比于曲率 R^{ab} 的贡献。回忆到 γ 矩阵满足如下的协变常数性质，就能很容易验证 (30) 等价于 (29) 的最后一项:

$$D_\mu \gamma^a = \partial_\mu \gamma^a + \omega_\mu{}^a{}_b \gamma^b + \frac{1}{4}\omega_\mu^{bc}\left[\gamma_{bc}, \gamma^a\right] = 0 \tag{31}$$

or that D_μ (scalar) = ∂_μ (scalar). From the definition of the covariant derivative acting on fermions, we find

或者 D_μ (标量) = ∂_μ (标量)。根据作用在费米子上的协变导数的定义，我们得到

$$\left[D_\mu, D_\upsilon\right] = \frac{1}{4}R_{\mu\upsilon}{}^{ab}\gamma_{ab}, \tag{32}$$

and therefore, using $\overline{\gamma_{ab}\psi_\rho} = -\overline{\psi}_\rho \gamma_{ab}$,

因此，利用 $\overline{\gamma_{ab}\psi_\rho} = -\overline{\psi}_\rho \gamma_{ab}$,

$$\overline{D_{[\mu}D_\upsilon \psi_{\rho]}} = -\frac{1}{8}R_{[\mu\upsilon}{}^{ab}\overline{\psi}_{\rho]}\gamma_{ab}. \tag{33}$$

Hence, the variation of the Rarita-Schwinger term becomes

于是，拉里塔-施温格项的变分为

$$\begin{aligned}\delta\mathcal{L}_{RS} &= -\frac{e}{16}M_P\overline{\psi}_\mu \gamma^{\mu\upsilon\rho}\gamma_{ab}\varepsilon R_{\upsilon\rho}{}^{ab} - \frac{e}{16}M_P\overline{\psi}_\rho \gamma_{ab}\gamma^{\mu\upsilon\rho}\varepsilon R_{\mu\upsilon}{}^{ab} + \ldots \\ &= -\frac{e}{16}M_P\overline{\psi}_\mu \left\{\gamma^{\mu\upsilon\rho}, \gamma_{ab}\right\}\varepsilon R_{\upsilon\rho}{}^{ab} + \ldots \\ &= -\frac{e}{2}M_P\overline{\psi}_\mu \gamma^\upsilon \varepsilon\left(R_\upsilon^\mu - \frac{1}{2}\delta_\upsilon^\mu R\right) + \ldots \\ &= -\frac{e}{2}M_P\overline{\psi}_\mu \gamma^\upsilon \varepsilon G_\upsilon{}^\mu + \ldots ,\end{aligned} \tag{34}$$

where we introduced the Einstein tensor $G_{\mu\nu}$ and used the identity $\left\{\gamma_{\mu\nu\rho}, \gamma^{ab}\right\} = -12\gamma_{[\mu}e_\upsilon^a e_{\rho]}^b$. The dots contain all terms that do not multiply the curvature of the spin connection. As expected, this can be compensated by

其中我们引入了爱因斯坦张量 $G_{\mu\nu}$ 并使用了恒等式 $\left\{\gamma_{\mu\nu\rho}, \gamma^{ab}\right\} = -12\gamma_{[\mu}e_\upsilon^a e_{\rho]}^b$ 。省略号包含了所有不与自旋联络曲率相乘的项。不出所料，这一项可以被下式抵消

$$\frac{\delta\mathcal{L}_{EH}}{\delta e_\mu^a}\delta e_\mu^a = -M_P^2 e e_a^\upsilon G_\upsilon{}^\mu \delta e_\mu^a, \tag{35}$$

which is also proportional to the same combination of the curvature, provided we define the variation of the vierbein as

只要我们将标架的变分定义为，它也正比于相同的曲率组合:

$$\delta e_\mu^a = \frac{1}{2M_P}\bar{\varepsilon}\gamma^a \psi_\mu \tag{36}$$

(recall that $\bar{\varepsilon}\gamma^a\psi_\mu = -\overline{\psi}_\mu\gamma^a\varepsilon$). Note that, proceeding in this way, we did not simply guess the variation of the vierbein from our considerations in the previous section, but instead really derived it. On the other hand, we see immediately that (36) is indeed consistent with (15).

(回忆 $\bar{\varepsilon}\gamma^a\psi_\mu = -\overline{\psi}_\mu\gamma^a\varepsilon$)。注意，按这种方式处理，我们不是从上一节的讨论直接猜出标架的变分，而是真正推导出来的。另一方面，我们可以立刻看出 (36) 确实与 (15) 自洽。

In order to complete the proof of the invariance of the action (23), we still need to discuss the following variations:

为了完成对作用量 (23) 不变性的证明，我们仍需要讨论以下变分:

(i) $\frac{\delta\mathcal{L}_{EH}}{\delta\omega_\mu^{ab}}\delta_\varepsilon\omega_\mu^{ab}$, (ii) $\frac{\delta\mathcal{L}_{RS}}{\delta\omega_\mu^{ab}}\delta_\varepsilon\omega_\mu^{ab}$; (iii) $\frac{\delta\mathcal{L}_{RS}}{\delta e_\mu^a}\delta_\varepsilon e_\mu^a$;

(i) $\frac{\delta\mathcal{L}_{EH}}{\delta\omega_\mu^{ab}}\delta_\varepsilon\omega_\mu^{ab}$, (ii) $\frac{\delta\mathcal{L}_{RS}}{\delta\omega_\mu^{ab}}\delta_\varepsilon\omega_\mu^{ab}$; (iii) $\frac{\delta\mathcal{L}_{RS}}{\delta e_\mu^a}\delta_\varepsilon e_\mu^a$;

(iv) Terms involving De^a from the partial integration in $\frac{\delta\mathcal{L}_{RS}}{\delta\psi_\mu}\delta_\varepsilon\psi_\mu$.

(iv) 来自 $\frac{\delta\mathcal{L}_{RS}}{\delta\psi_\mu}\delta_\varepsilon\psi_\mu$ 中分部积分、包含 De^a 的项。

We also need to understand and specify $\delta_\varepsilon\omega^{ab}$. As we will see, the variation of the spin connection will depend on the formalism (first, second, or 1.5 order) used to prove the invariance of the action.

我们还需要理解并确定 $\delta_\varepsilon\omega^{ab}$ 。我们将会看到，自旋联络的变分取决于证明作用量不变性时所采用的形式体系 (一阶、二阶或 1.5 阶)。

To do this calculation, we go back to the form expression (24). The variation of the action is then

为了进行这个计算，我们回到形式表达式 (24)。此时作用量的变分为

$$\begin{aligned}\delta S = &\underbrace{\frac{M_P^2}{4}D\delta\omega^{ab}\wedge e^c\wedge e^d\varepsilon_{abcd}}_{B1} + \underbrace{\frac{M_P^2}{2}R^{ab}\wedge\delta e^c\wedge e^d\varepsilon_{abcd}}_{A1}\\ &\underbrace{+\frac{i}{2}\delta e^a\wedge\overline{\psi}\wedge\gamma_5\gamma_a D\psi}_{B2} + \underbrace{\frac{i}{2}e^a\wedge\overline{\delta\psi}\wedge\gamma_5\gamma_a D\psi}_{A2}\\ &\underbrace{-\frac{i}{8}e^a\wedge\overline{\psi}\wedge\gamma_5\gamma_a\gamma_{cd}\psi\wedge\delta\omega^{cd}}_{B3} + \underbrace{\frac{i}{2}e^a\wedge\overline{\psi}\wedge\gamma_5\gamma_a D\delta\psi}_{A3},\end{aligned} \tag{37}$$

where the first line is the variation of S_{EH} and the last two lines come from varying S_{RS} , with $B3$ being due to the variation of the spin connection inside D .

其中第一行是 S_{EH} 的变分，最后两行来自对 S_{RS} 的变分，$B3$ 则来自 D 内部自旋联络的变分。

We know from previous computations that the term $A1$, coming from $\delta\mathcal{L}_{EH}/\delta e^a$, and the terms involving $D^2\varepsilon$ and $D^2\psi$ coming from $\delta\mathcal{L}_{RS}/\delta\psi$ cancel. In detail, the $D^2\varepsilon$ -term is $A3$, where one uses the explicit expression for $\delta\psi$, and the $D^2\psi$ -term can be extracted from $A2$ using the same steps that also led to (36). To do so, we switch the two-form $D\psi$ and the one-form $\delta\psi$, using (206), so that

由先前的计算我们可知，来自 $\delta\mathcal{L}_{EH}/\delta e^a$ 的项 $A1$ ，以及来自 $\delta\mathcal{L}_{RS}/\delta\psi$ 的含 $D^2\varepsilon$ 和 $D^2\psi$ 的项相互抵消。具体来说，利用 $\delta\psi$ 的显式表达式可得 $D^2\varepsilon$ 项为 $A3$ ，而 $D^2\psi$ 项可通过得到式 (36) 的相同步骤从 $A2$ 中提取。为此，我们利用式 (206) 交换二形式 $D\psi$ 和一形式 $\delta\psi$ ，得到

$$A2 \equiv \frac{i}{2}e^a \wedge \overline{\delta\psi} \wedge \gamma_5\gamma_a D\psi = \frac{i}{2}e^a \wedge \overline{D\psi} \wedge \gamma_5\gamma_a \delta\psi = \frac{i}{2}M_P e^a \wedge \overline{D\psi} \wedge \gamma_5\gamma_a D\varepsilon$$

(38)

and, integrating again by parts,

再次分部积分后，

(39)

$$A2 = -M_P d\left(\frac{i}{2}e^a \wedge \overline{D\psi}\gamma_5\gamma_a\varepsilon\right) + \underbrace{\frac{i}{2}M_P De^a \wedge \overline{D\psi}\gamma_5\gamma_a\varepsilon}_{A2''}$$

$$\underbrace{-\frac{i}{2}M_P e^a \wedge \overline{DD\psi}\gamma_5\gamma_a\varepsilon.}_{A2'}$$

The term $A2'$ then cancels $A1$ and $A3$ as before, and we are left with $\delta S = B1 + \ B2 + B3 + A2''$ plus boundary terms.

随后项 $A2'$ 与之前的 $A1$ 和 $A3$ 抵消，我们得到 $\delta S = B1 + \ B2 + B3 + A2''$ 加边界项。

To proceed further, we integrate by parts the term $B1$ and get

为进一步计算，我们对项 $B1$ 做分部积分，得到

$$B1 = \frac{M_P^2}{2}\delta\omega^{ab} \wedge De^c \wedge e^d \varepsilon_{abcd} + d\left(\frac{M_P^2}{4}\delta\omega^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}\right).$$

In order to write $B3$ in a very similar form, we can use properties of the γ -matrices

为了将 $B3$ 写成形式非常相似的表达式，我们可以利用 γ 矩阵的性质

$$\overline{\psi} \wedge \gamma_5\gamma_a\gamma_{cd}\psi = \overline{\psi} \wedge \gamma_5 \left(\gamma_{acd} + \eta_{ac}\gamma_d - \eta_{ad}\gamma_c\right)\psi = -i\overline{\psi} \wedge \gamma^e\psi\varepsilon_{acde} \tag{40}$$

so that, after some relabelling and reordering,

因此，重新标号和重排顺序后，

$$B3 = -\frac{1}{8}\delta\omega^{ab} \wedge \overline{\psi} \wedge \gamma^c\psi \wedge e^d\varepsilon_{abcd}. \tag{41}$$

Discarding boundary terms and inserting also (36) in B2, we then have

舍去边界项，并将式 (36) 代入 B2，我们得到

$$\delta S = \frac{M_P^2}{2}\delta\omega^{ab}\wedge\left(De^c - \frac{1}{4M_P^2}\overline{\psi}\wedge\gamma^c\psi\right)\wedge e^d\varepsilon_{abcd} \tag{42}$$

$$+\frac{i}{4M_P}\left(\bar{\varepsilon}\gamma^a\psi\right)\wedge\left(\overline{\psi}\wedge\gamma_5\gamma_a D\psi\right)+\frac{i}{2}M_P De^a\wedge\overline{D\psi}\gamma_5\gamma_a\varepsilon.$$

This expression can be simplified by rewriting the second line so that the torsion piece $\left(De^a - \frac{1}{4M_P^2}\overline{\psi}\wedge\gamma^a\psi\right)$ can also be factored out. While the last term of (42) obviously contains a derivative of the vielbein, the other term needs a reshuffling of the gravitini in order to produce the right bilinear without derivatives. We can achieve this by using the Fierz identity

该表达式可以通过改写第二行化简，从而也将挠率项 $\left(De^a - \frac{1}{4M_P^2}\overline{\psi}\wedge\gamma^a\psi\right)$ 提取出来。尽管式 (42) 的最后一项显然包含标架的导数，另一项需要重新排列引力微子的位置才能得到不含导数的正确双线性项。我们可以通过费尔兹恒等式完成这一点

$$\psi\wedge\overline{\psi} = \frac{1}{4}\left(\overline{\psi}\wedge\gamma^a\psi\right)\gamma_a - \frac{1}{8}\left(\overline{\psi}\wedge\gamma^{ab}\psi\right)\gamma_{ab} \tag{43}$$

and the fact that $\gamma^c\gamma^{ab}\gamma_c = 0$ and $\gamma^c\gamma^a\gamma_c = -2\gamma^a$:

以及 $\gamma^c\gamma^{ab}\gamma_c = 0$ 和 $\gamma^c\gamma^a\gamma_c = -2\gamma^a$ 满足的下述事实:

$$\frac{i}{4M_P}\left(\bar{\varepsilon}\gamma^a\psi\right)\wedge\left(\overline{\psi}\wedge\gamma_5\gamma_a D\psi\right) = -\frac{i}{8M_P}\left(\overline{\psi}\wedge\gamma_a\psi\right)\wedge\left(\overline{D\psi}\gamma_5\gamma^a\varepsilon\right). \tag{44}$$

Altogether, δS can then be written as

综上，δS 可写为

$$\delta S = \frac{M_P}{2}\left(De^a - \frac{1}{4M_P^2}\overline{\psi}\wedge\gamma^a\psi\right)\wedge\left[i\overline{D\psi}\gamma_5\gamma_a\varepsilon + M_P\delta\omega^{bc}\wedge e^d\varepsilon_{abcd}\right]$$

$$= \frac{M_P}{2}\left(De^a - \frac{1}{4M_P^2}\overline{\psi}\wedge\gamma^a\psi\right)\wedge\left[-\frac{1}{6}\overline{D\psi}\gamma^{bcd}\varepsilon + M_P\delta\omega^{bc}\wedge e^d\right]\varepsilon_{abcd},$$

(45)

where we have used $\gamma_5\gamma_a = (i/6)\,\varepsilon_{abcd}\gamma^{bcd}$. At this point, the variation of the spin connection assumes a primary role, and we can try to set (45) to zero in various different ways.

其中我们用到了 $\gamma_5\gamma_a = (i/6)\,\varepsilon_{abcd}\gamma^{bcd}$ 。至此，自旋联络的变分占据了核心地位，我们可以尝试通过多种不同方式令 (45) 等于零。

Second-Order Formalism

二阶形式主义

In this case, one imposes the so-called conventional constraint,

在该情形下，我们需要施加所谓的常规约束，

$$De^a = \frac{1}{4M_P^2}\overline{\psi} \wedge \gamma^a \psi \tag{46}$$

which determines the spin connection, $\omega_\mu^{ab} = \widehat{\omega}_\mu^{ab}(e,\psi)$, as the solution to this equation. The spin connection is thus treated from the very beginning as a dependent field, whose supersymmetry variation follows from the supersymmetry variations of e_μ^a and ψ_μ via the chain rule and the explicit functional dependence of $\widehat{\omega}_\mu^{ab}(e,\psi)$. By simple inspection of (45), however, we see that (46) already implies that $\delta_\varepsilon S = 0$, and we don’t really need to know $\delta_\varepsilon \omega_\mu^{ab}$.

该约束确定了自旋联络 $\omega_\mu^{ab} = \widehat{\omega}_\mu^{ab}(e,\psi)$ 为这个方程的解。因此自旋联络从一开始就被视为一个依赖场，它的超对称变换可通过链式法则以及 $\widehat{\omega}_\mu^{ab}(e,\psi)$ 的显式函数依赖关系，由 e_μ^a 和 ψ_μ 的超对称变换推导出。然而，通过直接观察 (45) 式我们可以发现，(46) 式已经给出 $\delta_\varepsilon S = 0$ ，我们并不需要真正知道 $\delta_\varepsilon \omega_\mu^{ab}$ 。

Let us nevertheless use the torsion

尽管如此，我们仍然使用挠率

$$T^a = \frac{1}{4M_P^2}\overline{\psi} \wedge \gamma^a \psi \tag{47}$$

to solve $De^a = T^a$ for the spin connection, which yields

来通过 $De^a = T^a$ 求解自旋联络，得到

$$\widehat{\omega}_\mu^{ab}(e,\psi) = \omega_\mu^{ab}(e) - \frac{1}{4M_P^2}\left(\overline{\psi}^{[a}\gamma_\mu \psi^{b]} - \overline{\psi}_\mu \gamma^{[a}\psi^{b]} - \overline{\psi}^{[a}\gamma^{b]}\psi_\mu\right), \tag{48}$$

where $\omega_\mu^{ab}(e)$ is the torsion-free spin connection (171) and the remaining piece is the contorsion tensor (183). This is used in the original approach of Ferrara, Freedman, and Van Nieuwenhuizen in [2]. It is interesting to point out that the supersymmetry variation of this connection, inherited from the variations of e_μ^a and ψ_μ , does not contain derivative terms, $\partial\varepsilon$, of the supersymmetry parameter:

其中 $\omega_\mu^{ab}(e)$ 是无挠自旋联络 (171)，剩余部分为反挠张量 (183)。这一方法被 Ferrara、Freedman 和 Van Nieuwenhuizen 用于文献 [2] 的原始工作中。值得指出的是，该联络由 e_μ^a 和 ψ_μ 的变换继承而来的超对称变换，并不包含超对称参数的导数项 $\partial\varepsilon$:

$$\delta\widehat{\omega}_\mu^{ab} = \frac{1}{M_P}\bar{\varepsilon}\gamma^\rho\left(D^\sigma\psi^\tau\right)\left(2e_\rho^{[a}e_\tau^{b]}g_{\mu\sigma} - e_\tau^{[a}e_\sigma^{b]}g_{\mu\rho}\right).$$

This is the reason why $\hat{\omega}_\mu^{ab}$ is often called supercovariant.

这就是为什么 $\hat{\omega}_\mu^{ab}$ 常被称为超协变的原因。

First-Order Formalism

一阶形式主义

This is the approach followed by Deser and Zumino in their original paper [3].

这是德泽尔 (Deser) 和祖米诺 (Zumino) 在他们的原始论文 [3] 中采用的方法。

Asking for the invariance of the action, $\delta S = 0$, via the vanishing of the term in square brackets in (45) fixes the variation of the spin connection to

根据 (45) 式中方括号项为零，要求作用量 $\delta S = 0$ 不变，即可将自旋联络的变分固定为

$$\delta\omega_\mu^{bc} = B_\mu^{bc} - \frac{1}{2}e_\mu^c B_e^{be} + \frac{1}{2}e_\mu^b B_e^{ce}, \tag{49}$$

with

其中

$$B_\mu^{bc} = \frac{i}{2M_P}\bar{\varepsilon}\gamma_\mu\gamma_5 D_\rho\psi_\sigma\varepsilon^{\rho\sigma bc}. \tag{50}$$

This can be extracted using the same trick that is used in Appendix ”Vierbein and Cartan’s Formalism” to derive the form (171) of the torsion-free spin connection in terms of the vierbein from the torsion constraint. It should be noted that (49) is not the same as the variation derived using second-order approach in the previous subsection. However, they become equivalent upon using the gravitino equations of motion.

我们可以利用“Vierbein(标架) 与嘉当形式主义”附录中，从挠率约束出发、用标架表示无挠自旋联络并得到形式 (171) 时所用的相同技巧来推导出这个结果。需要注意的是，(49) 式与上一小节用二阶方法导出的变分并不相同，但在运用引力微子运动方程后，二者等价。

1.5-Order Formalism

1.5 阶形式

In the 1.5-order formalism, one uses the fact that (46) can be obtained as a field equation from varying the action with respect to ω_μ^{ab} (as is obvious from the terms proportional to $\delta\omega^{bc}$ in (45)). Thus, when we determine the supersymmetry variation of the action and require ω_μ^{ab} to be determined by

在 1.5 阶形式中，我们利用如下事实: 式 (46) 可作为场方程，通过对 ω_μ^{ab} 变分作用量得到 (从式 (45) 中正比于 $\delta\omega^{bc}$ 的项可明显看出)。因此，当我们确定作用量的超对称变分，且要求 ω_μ^{ab} 由下式确定时

$$De^a = \frac{1}{4M_P^2}\bar{\psi}\wedge\gamma^a\psi \Leftrightarrow \frac{\delta S}{\delta\omega_\mu^{ab}} = 0,$$

we can immediately drop all terms proportional to $\delta\omega_\mu^{ab}$, as these are proportional to $\frac{\delta S}{\delta\omega_\mu^{ab}}$, which vanishes on-shell. Obviously, for the simple action we consider here, the only advantage over the second-order formalism is that we would not have to keep track of the $\delta\omega_\mu^{ab}$ terms in (45). Just as in the second-order formalism, the vanishing of the supersymmetry variation (45) is thus obtained by using (46), with the difference that (46) is now not imposed by hand, but arises as a field equation for the independent field ω_μ^{ab} . It is in this sense that the 1.5-order formalism combines elements from the first-order formalism (the a priori independence of the field ω_μ^{ab}) and from the second-order formalism (the use of (46) for the cancellation of (45)).

我们可以直接去掉所有正比于 δ $\delta\omega_\mu^{ab}$ 的项，因为这些项都正比于 $\frac{\delta S}{\delta\omega_\mu^{ab}}$ ，而 $\frac{\delta S}{\delta\omega_\mu^{ab}}$ 在壳上为零。显然，对于本文所讨论的简单作用量，该方法相较于二阶形式的唯一优势是我们无需追踪式 (45) 中的 $\delta\omega_\mu^{ab}$ 项。和二阶形式一样，超对称变分式 (45) 的零化同样可通过利用式 (46) 得到，区别在于此处式 (46) 不是手动施加的条件，而是独立场 ω_μ^{ab} 的场方程。正是在这个意义上，1.5 阶形式结合了一阶形式 (场 ω_μ^{ab} 先验独立) 和二阶形式 (利用式 (46) 抵消式 (45)) 的要素。

It should be stressed that the 1.5-order trick of using on-shell field equations in the supersymmetry variation can only be used for auxiliary fields such as ω_μ^{ab} .

需要强调的是，在超对称变分中利用在壳场方程的 1.5 阶技巧仅适用于像 ω_μ^{ab} 这样的辅助场。

Gauging the Poincaré Algebra

规范庞加莱代数

This section is reprinted from [1] When introducing general relativity as well as supergravity, we discussed the possibility of considering the vierbein and the spin connection as independent quantities. Do we have any conceptual reason behind this, in addition to the simplification of some computations? We will now see that an interesting perspective on gravity, which can help when dealing with supergravity, is that of considering gravity itself as a sort of a gauge theory where the gauge group is the Poincaré group [17, 18, 20] . This analogy will work only to a certain extent, but it will be very useful for understanding many specific new features that have to be introduced when one wants to promote supersymmetry to a local symmetry of nature. In fact, supergravity is the gauge theory of supersymmetry, and therefore, there must be a way to describe it as a theory where the gauge group is the Poincaré supergroup (or some other supergroup). For the sake of simplicity in this subsection, we set $M_P = 1$.

本节转载自 [1] ©2021 施普林格·自然旗下德国施普林格出版社，经授权重印，版权所有。在介绍广义相对论和超引力时，我们讨论过将标架和自旋联络视为独立量的可能性。除了简化部分计算外，这么做还有什么概念层面的依据吗？我们现在会看到，对引力有一个很有意思的视角，它在处理超引力时很有帮助: 即把引力本身看作一种规范理论，其中规范群是庞加莱群 [17, 18, 20] 。这个类比仅在一定范围内成立，但对于理解我们将超对称升级为自然界局域对称性时必须引入的诸多新特性，它非常有用。事实上，超引力就是超对称的规范理论，因此必然存在一种描述方式，将其表述为规范群是庞加莱超群 (或其他超群) 的理论。为了简化表述，本小节我们设定 $M_P = 1$ 。

Consider an ordinary gauge transformation $\delta_\varepsilon = \varepsilon^A T_A$, where T_A are the gauge generators satisfying

考虑一个普通规范变换 $\delta_\varepsilon = \varepsilon^A T_A$ ，其中 T_A 是满足下式的规范生成元

$$[T_A, T_B] = f_{AB}{}^C T_C,$$

with structure constants $f_{AB}{}^C$. If this is a global symmetry of an action, it can be made local by introducing vector fields, A_μ^A , for each symmetry so that the algebra

结构常数为 $f_{AB}{}^C$ 。如果这是作用量的整体对称性，我们可以为每个对称性引入向量场 A_μ^A 来将其局域化，使得代数

$$\left[\delta\left(\varepsilon_1^A\right), \delta\left(\varepsilon_2^B\right)\right] = \delta\left(\varepsilon_2^B \varepsilon_1^A f_{AB}{}^C\right) \tag{51}$$

with symmetry parameters ε_i^A has a faithful realization on them,

在对称参数为 ε_i^A 的情况下，能在这些场之上实现忠实表示，

$$\delta_\varepsilon A_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C A_\mu^B f_{BC}{}^A, \tag{52}$$

and we can introduce covariant derivatives

我们就可以引入协变导数

$$D_\mu = \partial_\mu - A_\mu^A T_A \tag{53}$$

acting non-trivially on fields which transform in non-trivial representations of the gauge group. The curvature, defined as

对在规范群非平凡表示下变换的场做非平凡作用。曲率定义为

$$\left[D_\mu, D_\upsilon\right] = -F_{\mu\upsilon}^A T^A \;\Leftrightarrow\; F_{\mu\upsilon}^A \equiv 2\partial_{[\mu} A_{\upsilon]}^A + A_\mu^B A_\upsilon^C f_{BC}{}^A, \tag{54}$$

transforms covariantly:

变换满足协变性:

$$\delta_\varepsilon F^A_{\mu\nu} = \varepsilon^C F^B_{\mu\nu} f_{BC}{}^A. \tag{55}$$

Let us now imagine that we want to make local the symmetries of the Poincaré group. The usual procedure is to introduce gauge fields in correspondence with the generators of the algebra. For the Poincaré algebra with generators P_a and M_{ab} , this means introducing two gauge fields, e^a_μ and ω^{ab}_μ , so as to match the gauge generators,

现在我们设想，我们想要让庞加莱群的对称性局域化。常规流程是对应代数的生成元引入规范场。对生成元为 P_a 和 M_{ab} 的庞加莱代数而言，这意味着引入两个规范场 e^a_μ 和 ω^{ab}_μ ，以匹配规范生成元，

$$A^A_\mu T_A = e^a_\mu P_a + \frac{1}{2}\omega^{ab}_\mu M_{ab}. \tag{56}$$

Given the particular form of the Poincaré algebra, the gauge curvatures of these vectors are precisely

根据庞加莱代数的特殊形式，这些向量的规范曲率恰好就是

$$T^a = de^a + \omega^a{}_b \wedge e^b \tag{57}$$

and

和

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{58}$$

where the spin connection and the vierbein are so far independent fields.

其中自旋联络和标架目前是相互独立的场。

This construction is perfectly legitimate. However, it clearly leads to an ordinary gauge theory (Due to the Poincaré algebra being non-semisimple and non-compact, the standard kinetic terms of the gauge fields would not be positive definite so that this would actually not be a unitary theory.) and not to a gravity theory as we would like. From the T^a and R^{ab} curvatures, we could construct kinetic terms giving the propagation of independent degrees of freedom and discuss the resulting gauge theory, where the Poincaré group is realized on the vector fields as

这个构造完全合理。但它显然只能得到普通规范理论 (由于庞加莱代数是非半单、非紧李代数，规范场的标准动能项不是正定的，因此这实际上不是幺正理论)，而非我们想要的引力理论。我们可以从 T^a 和 R^{ab} 曲率出发，构造出独立自由度传播的动能项，讨论由此得到的规范理论，其中庞加莱群按如下形式作用在向量场上

$$\delta_P \omega^{ab} = 0, \tag{59}$$

$$\delta_P e^a = D\varepsilon^a, \tag{60}$$

$$\delta_M \omega^{ab} = d\Lambda^{ab} - \omega^a{}_c \Lambda^{cb} - \Lambda^{ac} \omega_c{}^b, \tag{61}$$

$$\delta_M e^a = \Lambda^a{}_c e^c, \tag{62}$$

with gauge parameters ε^a and Λ^{ab} . If, on the other hand, we want to get only the metric degrees of freedom, we have to impose a constraint between ω^{ab} and e^a . The constraint that does this job is the conventional constraint or torsion constraint (see [19, 20] for details),

> 规范参数为 ε^a 和 Λ^{ab} 。另一方面，如果我们只想得到度规自由度，就必须对 ω^{ab} 和 e^a 施加约束。实现这一点的是常规约束，或称挠率约束 (详见 [19, 20]),

$$T^a = 0\,. \tag{63}$$

This constraint, however, is not invariant under (59)-(60):

> 但该约束在 (59)-(60) 下不具有不变性:

$$\delta_P T^a = \delta_P D e^a = D \delta_P e^a = DD\varepsilon^a = -R^a{}_b \varepsilon^b \neq 0. \tag{64}$$

This means that if we impose the conventional constraint, translation symmetry is broken. Moreover, it is also clear that now the spin connection ω^{ab} cannot be treated as independent of the vielbein anymore and hence the transformation (59) will no longer be valid. Indeed, since $\omega^{ab} = \omega^{ab}(e)$, the spin connection is not invariant under translations

> 这意味着如果我们施加常规约束，平移对称性就会被破坏。此外，很明显此时自旋联络 ω^{ab} 不能再看作独立于标架的量，因此变换 (59) 不再成立。事实上，由于 $\omega^{ab} = \omega^{ab}(e)$ ，自旋联络在平移下不具有不变性

$$\delta_P \omega^{ab} = \int d^4x \frac{\delta \omega^{ab}}{\delta e^c} \delta_P e^c \neq 0.$$

The final outcome of this discussion is that, when the conventional constraint is imposed, the Poincaré gauge algebra is deformed and translational symmetry is replaced by a new invariance under diffeomorphisms. This can be seen by considering the commutator of two translation generators on the vierbein:

> 本次讨论的最终结论是: 当施加常规约束时，庞加莱规范代数会发生形变，平移对称性被微分同胚下的新不变性取代。这一点可以通过研究双标架上两个平移生成元的对易子看出:

$$[\delta_{P2}, \delta_{P1}]\, e^a = -\delta_{P[2}\left(D\varepsilon^a_{1]}\right) = -\delta_{P[2}\omega^a{}_c \varepsilon^c_{1]}, \tag{65}$$

and now $\delta_P \omega^{ab} \neq 0$. The resulting algebra then has a non-vanishing commutator

> 现在得到 $\delta_P \omega^{ab} \neq 0$ 。最终得到的代数具有非零对易子

$$[P,P] \neq 0,$$

as is appropriate for general coordinate transformations, which do not commute. Actually, one can check that using the constraint (63), the translation generators on the vielbein take the form of general coordinate transformations.

这符合非对易的广义坐标变换的性质。实际上，可以验证，利用约束 (63)，标架上的平移生成元取广义坐标变换的形式。

The action constructed from the curvatures and the vierbein then is invariant with respect to local Lorentz transformations and diffeomorphisms. The infinitesimal change of a function under a diffeomorphism is given by the Lie derivative L_ε , and therefore, the action is going to be invariant if (cf. Appendix "Lie Derivative on P-Forms")

由曲率和双标架构造的作用量在局域洛伦兹变换和微分同胚下保持不变。微分同胚下函数的无穷小变化由李导数 L_ε 给出，因此作用量满足以下条件时保持不变 (参见附录“p-形式的李导数”):

$$L_\varepsilon S = \int d\left(\iota_\varepsilon \mathcal{L}\right) + \int \iota_\varepsilon d\mathcal{L} = 0,$$

but the first term is a total derivative that can be discarded, while the second is zero because $d\mathcal{L}$ has one degree more than the top form. Finally, in the construction of an action, we will not make use of a kinetic term of the form $R^{ab} \wedge \star R_{ab}$ because of the conventional constraint which makes it quartic in the derivatives. The appropriate quadratic term is the Einstein-Hilbert action above.

但第一项是全导数，可以舍去；第二项为零，因为 $d\mathcal{L}$ 比最高形式高一阶。最后，在构造作用量时，我们不会使用 $R^{ab} \wedge \star R_{ab}$ 形式的动能项，因为常规约束会使其导数部分成为四次项。合适的二次项就是上述的爱因斯坦-希尔伯特作用量。

The method we have outlined in this section can be easily extended to generic supergravity theories by extending the Poincaré algebra to the super-Poincaré algebra by including fermionic generators and possibly other bosonic generators for the internal symmetries. The power of this approach lies in the ease of guessing the transformation laws under the various symmetries, including supersymmetry. This means that this approach can be used as a guide to derive and construct the Lagrangian and/or the equations of motion of systems respecting any symmetry group we would like to realize. Once again, we stress that one has to be careful with its application because of the constraints that will be needed to obtain a consistent gravity theory (invariant under diffeomorphisms). Imposing these constraints will break the transformation rules that do not preserve them.

本节概述的方法可以很容易推广到一般超引力理论: 通过引入费米生成元以及内部对称性可能存在的其他玻色生成元，将庞加莱代数扩充为超庞加莱代数即可。这种方法的优势在于，它可以很方便地得到包括超对称在内不同对称性下的变换规律。这意味着该方法可以作为推导和构造满足任意期望实现的对称群的系统的拉格朗日量和/或运动方程的指导。我们再次强调，应用该方法时必须谨慎，因为得到一致的 (在微分同胚下不变的) 引力理论需要引入约束，施加这些约束会破坏不满足约束的变换规则。

We end this discussion with a few remarks on the gauging of the super-Poincaré group. We could gauge this algebra by adding new vector fields $\psi_{\mu A}$ for the fermionic generators Q^A . From the algebra, we then have

我们在讨论的最后对超庞加莱群的规范场化做几点说明。我们可以通过为费米生成元 Q^A 引入新的矢量场 $\psi_{\mu A}$ 来对该代数做规范场化，根据代数可得

$$A_\mu^A T_A = e_\mu^a P_a + \frac{1}{2}\omega_\mu^{ab} M_{ab} + \overline{\psi}_{\mu A} Q^A + \overline{\psi}_\mu^A Q_A, \tag{66}$$

and we can read the supersymmetry transformations by applying

我们可以通过应用下述方法读出超对称变换:

$$\delta_\varepsilon A_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C A_\mu^B f_{BC}{}^A.$$

For instance, for $\mathcal{N} = 1$ supergravity, we would get that the spin connection is invariant,

例如，对于 $\mathcal{N} = 1$ 超引力，我们可以得到自旋联络是不变的,

$$\delta_\varepsilon \omega^{ab} = 0, \tag{67}$$

because the Lorentz generator never appears on the right-hand side of any commutator involving the supersymmetry generator. However, just like for the bosonic case, we should impose a torsional constraint in order for the vierbein and spin connection not to be independent. Doing so, we would fix the form of the spin connection as $\omega^{ab} = \omega^{ab}(e, \psi)$ and could check the new realization of the algebra on the fields.

这是因为洛伦兹生成元从不出现在任何包含超对称生成元的对易子的右侧。但和玻色子的情况一样，我们需要施加挠率约束，使得双标架和自旋联络不对易。施加该约束后，我们可以将自旋联络的形式固定为 $\omega^{ab} = \omega^{ab}(e, \psi)$，并可以验证该代数在场上的新实现。

One last interesting remark involves the definition of the gauge curvatures for the super-Poincaré algebra. From the structure constants of the supersymmetry algebra, one can deduce a new definition for the curvatures, including the one of the translation generators, which is usually denoted as the two-form $\mathcal{T}^a$. Since the translation generators P_a appear on the right-hand side of the commutator of two supercharges, the corresponding curvature definition is now

最后一个有趣的结论涉及超庞加莱代数规范曲率的定义。根据超对称代数的结构常数，我们可以推导出曲率的新定义，包括平移生成元对应的曲率，它通常被记为二次型 $\mathcal{T}^a$ 。由于平移生成元 P_a 出现在两个超荷对易子的右侧，因此对应的曲率定义现在写为

$$\mathcal{T}^a = De^a - \frac{1}{4}\overline{\psi}^A \gamma^a \psi_A \tag{68}$$

and involves a fermion bilinear. This means that imposing the constraint $\mathcal{T}^a = 0$ results in a spin connection depending on the gravitino fields. Hence, supergravity is often referred to as a theory with non-trivial torsion for the spin connection, because $\mathcal{T}^a = 0$ implies $De^a \neq 0$.

并且包含一个费米子双线性项。这意味着施加约束 $\mathcal{T}^a = 0$ 会让自旋联络依赖于引力微子场。因此，超引力通常被认为是自旋联络具有非平凡挠率的理论，因为 $\mathcal{T}^a = 0$ 蕴含 $De^a \neq 0$ 。

Adding a Cosmological Constant

添加宇宙学常数

 So far, we considered the construction of a supergravity action around a Minkowski background, whose non-linear completion led to Einstein gravity coupled to a gravitino field without a cosmological constant. In ordinary Einstein gravity, however, we can always add a cosmological constant Λ to obtain the action

到目前为止，我们讨论了闵氏背景下超引力作用量的构造，其非线性完备化得到了耦合引力微子场且不带宇宙学常数的爱因斯坦引力。但在普通爱因斯坦引力中，我们总可以添加宇宙学常数 Λ，得到如下作用量

$$S = M_P^2 \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \Lambda\right) \tag{69}$$

and find a maximally symmetric vacuum with

并由此找到一个具有下述性质的极大对称真空

$$\Lambda > 0 \leftrightarrow \text{ de Sitter (dS)},$$

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{70}$$

$$\Lambda < 0 \leftrightarrow \text{ Anti-de Sitter (AdS)}.$$

It is natural to ask whether these solutions and the corresponding actions can be supersymmetrized in a natural way. In this section, we will focus on pure supergravity theories (without matter multiplets). If we want to construct a supergravity action generalizing (69), we should be able to find a supergroup that contains the symmetry group of AdS and/or dS spacetime. We will now see that minimal supersymmetry constrains the closure of the algebra in a way that only one of the two options is consistent (There are consistent de Sitter superalgebras with extended supersymmetries, but they do not allow for positive weight representations, and hence, their realizations have the wrong sign in front of the kinetic terms of some of their fields [21,22].)

我们自然会问，这些解和对应的作用量能否以自然的方式实现超对称化。本节我们将聚焦于纯超引力理论 (不含物质多重态)。如果我们要构造推广 (69) 式的超引力作用量，应当能找到一个包含反德西特 (AdS) 和/或德西特 (dS) 时空对称群的超群。我们会看到，最小超对称性对代数的封闭性给出约束，使得两种选择中只有一种自洽 (确实存在扩展超对称下自洽的德西特超代数，但它们不允许正权表示，因此其实现中部分场动能项前的符号错误 [21,22]。)

Before discussing the corresponding superalgebras, we note that the symmetry groups of both AdS and dS in d dimensions, $\mathrm{SO}(1,d)$ and $\mathrm{SO}(2,d-1)$, respectively, can be embedded in $\mathrm{SO}(2,d)$. For $d=4$, the (A)dS algebra is described by ten anti-Hermitian generators $M_{\underline{AB}}$ satisfying the commutator relations

在讨论对应的超代数之前，我们指出，AdS 和 dS 在 d 维的对称群分别为 $\mathrm{SO}(1,d)$ 和 $\mathrm{SO}(2,d-1)$ ，二者都可以嵌入 $\mathrm{SO}(2,d)$ 。对于 $d=4$ ，(A)dS 代数由十个反埃尔米特生成元 $M_{\underline{AB}}$ 描述，它们满足如下对易关系

$$\left[M_{\underline{AB}},M_{\underline{CD}}\right]=-2\eta_{\underline{C}[\underline{A}}M_{\underline{B}]\underline{D}}+2\eta_{\underline{D}[\underline{A}}M_{\underline{B}]\underline{C}}, \tag{71}$$

with $\eta_{AB}=\mathrm{diag}\{-+++-\}$ for AdS and $\eta_{AB}=\mathrm{diag}\{-++++\}$ for dS space. The explicit (A)dS algebra follows by identifying $M_{5a}=\ell P_a$, where we split $\underline{A}=\{a,5\}$, with $a,b,\ldots=0,1,2,3$, and ℓ is the radius of curvature of (A)dS:

其中 AdS 对应 $\eta_{AB}=\mathrm{diag}\{-+++-\}$ ，dS 空间对应 $\eta_{AB}=\mathrm{diag}\{-++++\}$ 。通过标识 $M_{5a}=\ell P_a$ 可以得到显式的 (A)dS 代数，这里我们拆分 $\underline{A}=\{a,5\}$ ，取 $a,b,\ldots=0,1,2,3$ ，而 ℓ 是 (A)dS 的曲率半径:

$$\begin{gathered}[M_{ab},M_{cd}]=-2\eta_{c[a}M_{b]d}+2\eta_{d[a}M_{b]c},\\ [P_a,M_{bc}]=2\eta_{a[b}P_{c]},\\ [P_a,P_b]=\pm\frac{1}{\ell^2}M_{ab},\end{gathered} \tag{72}$$

where the last commutator is equivalent to $[M_{5a},M_{5b}]=-\eta_{55}M_{ab}$ and $\eta_{55}=-1$ for AdS space and $\eta_{55}=+1$ for dS. Hence, the upper sign is for AdS and the lower one for dS spacetime. Clearly, when $\ell\to\infty$, the (A)dS curvature goes to zero, and one gets back the Poincaré algebra.

其中最后一个对易子，AdS 空间等价于 $[M_{5a},M_{5b}]=-\eta_{55}M_{ab}$ 和 $\eta_{55}=-1$ ，dS 等价于 $\eta_{55}=+1$ 。因此上符号对应 AdS，下符号对应 dS 时空。显然，当 $\ell\to\infty$ 时，(A)dS 曲率趋于零，我们回到庞加莱代数。

To construct the full superalgebra, one needs to specify also the commutators with the supercharges. In particular, $[P_a,Q]$ cannot be zero anymore, as it used to be in the super-Poincaré case, because we would no longer close the super-Jacobi identities, as

要构造完整的超代数，还需要指定和超荷的对易子。特别地，$[P_a,Q]$ 不能再像超庞加莱情形中那样等于零，因为那样超雅可比恒等式将无法封闭，原因是

$$\left[\underbrace{[P_a,P_b]}_{\sim M_{ab}},Q\right]+\left[\underbrace{[P_a,Q]}_{0},P_b\right]-\left[\underbrace{[P_b,Q]}_{0},P_a\right]=0. \tag{73}$$

We therefore need to impose new commutator relations such that the momenta do not commute with the supercharges. To respect Lorentz covariance and the graded algebra structure, the result of the commutator should be proportional to the supercharges and come with some gamma matrices. In principle, there are two possibilities that respect the Majorana condition on the supercharges

因此我们需要引入新的对易关系，让动量不再和超荷对易。为了满足洛伦兹协变性和分次代数结构，对易子的结果应当正比于超荷，并带有伽马矩阵。原则上，有两种满足超电荷马约拉纳条件的可能形式

$$[P_a, Q] \sim \gamma_a Q, \text{ or } [P_a, Q] \sim \gamma_a \gamma_5 Q. \tag{74}$$

In the first case, the coefficient multiplying the right-hand side should be real, while in the second, it should be imaginary. If we use a chiral notation, the sign and the ambiguities can be reabsorbed in a single dimensionful complex coefficient $\tilde{g}$. We stress this fact, because there are sometimes wrong statements in the literature about this. Once we introduce the chiral notation, the new commutators are

第一种情形下，右侧的系数应为实数，第二种情形下应为虚数。如果采用手征记号，符号和不确定性都可以被吸收进单个量纲为复系数的 $\tilde{g}$ 中。我们需要强调这一点，因为文献中有时会对此存在错误论述。引入手征记号后，新的对易子为

$$[P_a, Q_R] = -\frac{\tilde{g}}{2}\gamma_a Q_L, \; [P_a, Q_L] = -\frac{\tilde{g}^*}{2}\gamma_a Q_R. \tag{75}$$

Once we introduce these new non-trivial commutators, the super-Jacobi identity can be satisfied, though only for the AdS case. This is readily seen by explicitly computing the results of the various commutators:

引入这些新的非平凡对易子后，超雅可比恒等式可以满足，但仅对 AdS 情形成立。直接显式计算各类对易子的结果就能看出这一点:

$$\begin{aligned} 0 &\stackrel{!}{=} [P_a, [P_b, Q_L]] + [Q_L, [P_a, P_b]] - [P_b, [P_a, Q_L]] \\ &= \frac{|g|^2}{4}(\gamma_b\gamma_a - \gamma_a\gamma_b) Q_L \pm \frac{1}{\ell^2}[Q_L, M_{ab}] \\ &= -\frac{|g|^2}{2}\gamma_{ab} Q_L \pm \frac{1}{2\ell^2}\gamma_{ab} Q_L, \end{aligned} \tag{76}$$

where we used the commutation relations in (217), but with the anti-Hermitian generators $P_a = i\mathcal{P}_a$ and $M_{ab} = i\mathcal{M}_{ab}$. It is now clear that only for the plus sign we can get a solution:

此处我们使用了式 (217) 中的对易关系，但其中生成元 $P_a = i\mathcal{P}_a$ 和 $M_{ab} = i\mathcal{M}_{ab}$ 为反厄米的。显然只有取正号时我们才能得到解:

$$|g|^2 = \frac{1}{\ell^2} \tag{77}$$

Hence, only for AdS we can write a consistent supersymmetric completion with a single supercharge. We finally note that the closure of the super-Jacobi identities requires that another commutator gets modified, namely,

因此，仅在反德西特空间中，我们才能写出带单个超荷的自洽超对称完备化。最后我们指出，超雅可比恒等式的闭包要求另一个对易子发生修改，即:

$$\{Q_L, \bar{Q}_L\} = \tilde{g}^* \gamma^{ab} M_{ab}. \tag{78}$$

The constraint imposing that the superalgebra can be defined only for the AdS supergroup and not for dS implies a very important fact: a positive cosmological constant will always break supersymmetry, while a negative cosmological constant may be compatible with supersymmetry.

该约束要求超代数仅对反德西特超群成立，不对德西特超群成立，这引出了一个非常重要的结论: 正宇宙学常数总会破坏超对称，而负宇宙学常数可以与超对称性相容。

Although matter couplings or extended supersymmetries may allow for de Sitter vacua in a supersymmetric theory, the vacuum itself will always break supersymmetry.

尽管物质耦合或扩展超对称可能允许超对称理论中存在德西特真空，但真空本身始终会破坏超对称。

Once supersymmetry is broken, one could describe this phase of the theory by using non-linear realizations, as it is customary for any other symmetry whose linear action is broken. This has been the subject of intense scrutiny (see, for instance, [23-26]), and one can indeed write actions for theories with dS vacua where supersymmetry is non-linearly realized. Since an effective discussion of this topic requires some additional technical introduction to superfields in supergravity, we will not deal with it here, but refer the reader to the literature on the subject, such as [9].

一旦超对称被破坏，我们就可以像处理其他任何线性作用被破坏的对称性那样，用非线性实现来描述理论的这一相。这一课题已经得到了大量研究 (例如参见文献 [23-26])，我们确实可以为超对称非线性实现的德西特真空理论写出作用量。由于对该主题的有效讨论需要额外补充介绍超引力中超场的相关技术，我们在此不展开讨论，读者可以参考该主题的相关文献，例如文献 [9]。

Construction of the Action

作用量的构造

Now that we established that the anti-de Sitter group can be consistently extended to a supergroup, we would like to realize it in terms of a supersymmetric action that includes a negative cosmological constant. We will proceed in a fashion similar to what has been done in the flat case, starting from the supersymmetry transformation of the gravitino and then trying to close the action of the supersymmetry transformation on the free Lagrangian for the gravity multiplet, possibly introducing interaction terms. We therefore need to fix first the supersymmetry transformation rule of the gravitino. Since the algebra has been modified with respect to the case without cosmological constant, we expect that also the supersymmetry transformations get modified accordingly.

既然我们已经证明反德西特群可以一致地推广为超群，我们希望通过包含负宇宙学常数的超对称作用量来实现它。我们的构造过程与平直时空情形的处理类似：从引力微子的超对称变换出发，尝试在引力多重态的自由拉格朗日量上封闭超对称变换的代数，必要时引入相互作用项。因此我们首先需要确定引力微子的超对称变换规则。由于该代数与无宇宙学常数情形相比已经发生了修改，我们预期超对称变换也会相应发生修改。

As sketched in section ”Gauging the Poincaré Algebra,” we can generically deduce the supersymmetry transformation properties of the various fields from the structure constants of the underlying superalgebra. For supergravity without cosmological constant, one would do this by viewing supergravity as a gauge theory of the super-Poincaré group, but with some constraints needed to relate the vierbein and the spin connection degrees of freedom.

正如“规范庞加莱代数”一节所述，我们一般可以从基础超代数的结构常数推导出各场的超对称变换性质。对于无宇宙学常数的超引力，我们可以将其看作超庞加莱群的规范理论，只需要引入一些约束来联系 vierbein 与自旋联络的自由度。

By using this trick, we can now deduce the supersymmetry transformation of the gravitino in the presence of a negative cosmological constant by looking at the structure constants of the AdS superalgebra coming from commutators which have a supersymmetry generator on the right-hand side. This inspection shows that a new term in the supersymmetry transformation of the gravitino should appear because of the non-zero commutator (75) between the translation generators and supersymmetry generators. If one interprets the spin connection term in the Lorentz-covariant derivative in the original gravitino transformation (26) as due to the non-vanishing commutator of M_{ab} with Q , the new non-vanishing commutator (75) between P_a and Q should then analogously lead to an additional contribution to the gravitino transformation so as to make the transformation covariant with respect to the full AdS isometry group. In $\delta\psi_{\mu L}$, this additional contribution should be of the form given in the first equation of (75) contracted with the gauge field of the translation generator P_a , i.e., with the vierbein e^a_μ . We therefore should have

利用这一思路，我们现在可以通过分析 AdS 超代数中，右侧为超对称生成元的对易子的结构常数，推导出存在负宇宙学常数时引力微子的超对称变换。分析表明，由于平移生成元和超对称生成元之间存在非零对易子 (75)，引力微子的超对称变换中应当出现一个新项。如果我们把原引力微子变换 (26) 中洛伦兹协变导数里的自旋联络项，解释为来自 M_{ab} 与 Q 的非零对易子，那么 P_a 与 Q 之间新的非零对易子 (75) 也应当类似地对引力微子变换贡献一项额外项，使得变换在全反德西特等距群下协变。在 $\delta\psi_{\mu L}$ 中，这一额外项应当满足形式：由 (75) 的第一个方程与平移生成元 P_a 的规范场，即 vierbein e^a_μ 缩并得到。因此我们有

$$\delta\psi_{\mu L} = M_P D_\mu \varepsilon_L - \frac{g}{2} M_P^2 \gamma_\mu \varepsilon_R, \tag{79}$$

where now we have a dimensionless constant $g \in \mathbb{C}$, because of the introduction of the dimensionful M_P and M_P^2 factors. Clearly, this has to go together with the conjugate relation:

其中由于引入了量纲为 M_P 和 M_P^2 的因子，现在这里出现了一个无量纲常数 $g \in \mathbb{C}$。显然，这必须与共轭关系共存：

$$\delta\psi_{\mu R} = M_P D_\mu \varepsilon_R - \frac{g^*}{2} M_P^2 \gamma_\mu \varepsilon_L. \tag{80}$$

Once again, we stress that g here can be any complex number, because there are sometimes wrong statements in the literature about this.

我们再次强调，此处 g 可以是任意复数，因为文献中有时会出现关于这一点的错误论断。

To construct the action, we start from the action (23) with the vierbein transformation rule

为了构造作用量，我们从满足 vierbein 变换规则的作用量 (23) 出发

$$\delta e_\mu^a = \frac{1}{2M_P} \bar{\varepsilon}_L \gamma^a \psi_{\mu R} + \text{ h.c.}$$

and (79) for the gravitino

而引力微子满足 (79)

$$\delta\psi_{\mu L} = M_P D_\mu \varepsilon_L - \frac{g}{2} M_P^2 \gamma_\mu \varepsilon_R.$$

The reason we start from the action without the cosmological constant, rather than adding explicitly the cosmological constant among the bosonic terms right from the beginning, is that it will automatically be enforced by supersymmetry in an iterative procedure at higher order in g , as will become clear momentarily.

我们从无宇宙学常数的作用量出发，而非一开始就直接在玻色项中加入宇宙学常数的原因是，正如我们很快就会看到的，宇宙学常数会在 g 高阶项的迭代过程中被超对称自动要求出现。

The gravitino relation differs from the one in (26) by a shift term proportional to the constant g . Clearly, this shift breaks the supersymmetry of the original action (23), and we need to restore it by adding additional terms to it. In the following, we will establish again the invariance under supersymmetry of a modified action. To our knowledge, this was first done in [27].

引力微子的关系与 (26) 相比，多了一个正比于常数 g 的平移项。显然，这个平移项会破坏原作用量 (23) 的超对称性，我们需要通过给作用量添加额外项来恢复超对称。接下来我们将重新证明修改后的作用量具有超对称不变性。据我们所知，这一工作最早由文献 [27] 完成。

The first supersymmetry-breaking effect of the shift term (proportional to g) is that of generating new terms in the variation of the Rarita-Schwinger part of the Lagrangian. To compute these terms, we use the supersymmetry variation of the conjugate gravitino, which, in form notation, reads

这个平移项 (正比于 g) 造成的第一个超对称破缺效应，是会在拉格朗日量的拉里塔-施温格部分变分时产生新项。为了计算这些项，我们使用共轭引力微子的超对称变分，其形式记号下的表达式为

$$\delta\overline{\psi}_R = M_P \overline{D\varepsilon}_R + \frac{g^*}{2} M_P^2 \bar{\varepsilon}_L \gamma_a e^a, \tag{81}$$

as one may easily verify. Denoting by δ_g the variations due to the $\mathcal{O}(g)$ shift term in the gravitino transformation law, the uncancelled variation of $\mathcal{L}_{RS}$ under supersymmetry is then

这很容易验证。将引力微子变换律中由 $\mathcal{O}(g)$ 平移项带来的变分记作 δ_g ，那么 $\mathcal{L}_{RS}$ 在超对称下未被抵消的变分就是

$$\delta_g \mathcal{L}_{RS} = \frac{i}{2} e^a \wedge \delta_g \overline{\psi}_R \wedge \gamma_5 \gamma_a D\psi_L + \frac{i}{2} e^a \wedge \overline{\psi}_L \wedge \gamma_5 \gamma_a D \delta_g \psi_R + \text{ h.c.}$$

$$= \frac{i}{4} g^* M_P^2 e^a \wedge e^b \wedge \bar{\varepsilon}_L \gamma_b \gamma_5 \gamma_a D\psi_L$$

$$- \frac{i}{4} g^* M_P^2 e^a \wedge \overline{\psi}_L \wedge \gamma_5 \gamma_a D \left(e^b \gamma_b \varepsilon_L \right) + \text{ h.c.}$$

$$= \frac{i}{4} g^* M_P^2 e^a \wedge e^b \wedge \bar{\varepsilon}_L \gamma_5 \gamma_{ab} D\psi_L - \frac{i}{4} g^* M_P^2 e^a \wedge De^b \wedge \overline{\psi}_L \gamma_5 \gamma_a \gamma_b \varepsilon_L$$

$$- \frac{i}{4} g^* M_P^2 e^a \wedge e^b \wedge \overline{\psi}_L \wedge \gamma_5 \gamma_{ab} D\varepsilon_L + \text{ h.c.}$$

(82)Integrating by parts the first term in the last equality, we get

对最后一个等式的第一项分部积分后，我们得到

$$\delta_g \mathcal{L}_{RS} = \frac{i}{2} g^* M_P^2 e^a \wedge De^b \wedge \left(\bar{\varepsilon}_L \gamma_5 \gamma_{ab} \psi_L - \frac{1}{2} \overline{\psi}_L \gamma_5 \gamma_a \gamma_b \varepsilon_L \right)$$

$$- \frac{i}{2} g^* M_P^2 e^a \wedge e^b \wedge \overline{\psi}_L \wedge \gamma_5 \gamma_{ab} D\varepsilon_L + \text{ h.c.} \tag{83}$$

The first term, proportional to De^a , plays a similar role as in the case without a cosmological constant and will be discussed later after Equation (90). Since the remaining terms are proportional to the derivative of the supersymmetry parameter, we can try and use the supersymmetry transformation rule of the gravitini, $\delta\psi_L = M_P D\varepsilon_L + \mathcal{O}(g)$, to cancel them. For this reason, we add a mass-like term to the Lagrangian:

第一项与 De^a 成正比，其作用与无宇宙常数情形类似，我们将在 (90) 式之后再讨论。由于其余项正比于超对称参数的导数，我们可以利用引力微子的超对称变换规则 $\delta\psi_L = M_P D\varepsilon_L + \mathcal{O}(g)$ 来抵消这些项。为此，我们给拉格朗日量添加一个类质量项:

$$\mathcal{L}_{\mathcal{M}_\psi} = \frac{i}{4} g^* M_P e^a \wedge e^b \wedge \overline{\psi}_L \wedge \gamma_5 \gamma_{ab} \psi_L + \frac{i}{4} g M_P e^a \wedge e^b \wedge \overline{\psi}_R \wedge \gamma_5 \gamma_{ab} \psi_R, \tag{84}$$

so that the variation of ψ in (84) compensates for (83) at order g . We point out here the reduced Planck mass factors, so that the whole coefficient has mass dimension 1, as well as the overall factor 1/4, which is due to the double variation required to match (83).

这样 (84) 式中 ψ 的变分可以在 g 阶抵消 (83) 式。我们在此指出约化普朗克质量因子，使得整个系数的质量量纲为 1，另外还有整体因子 1/4，该因子源于匹配 (83) 所需的双重变分。

While the introduction of (84) allows the cancellation of the $D\varepsilon$ terms in (83), it also gives rise to two further new variations we have to take care of. One variation comes at order g from the variation of the vierbein in (84). We will discuss its cancellation together with the cancellation of the De^a terms in (83) further below. The other new variation of (84) is of order g^2 and arises when the order g shift term in the gravitino transformation is used in the variation of the gravitini in (84). Indeed, the order g^2 variation of the gravitino mass term produces (suppressing the wedges)

引入 (84) 式虽然可以抵消 (83) 式中的 $D\varepsilon$ 项，但同时还会产生另外两个我们需要处理的新增变分。一个变分来自 (84) 式中 vierbein 的变分，处于 g 阶，我们将在下文讨论它和 (83) 式中 De^a 项的抵消。(84) 式的另一个新增变分处于 g^2 阶，它来自对 (84) 式中引力微子变分时，用到了引力微子变换中的 g 阶平移项。实际上，引力微子质量项的 g^2 阶变分给出 (省略外积楔符号):

$$\delta_g \mathcal{L}_{\mathcal{M}_\psi} = -\frac{i}{8}|g|^2 M_P^3 e^a e^b \left(\overline{\psi}_L \gamma_5 \gamma_{ab} e^c \gamma_c \varepsilon_R - \overline{\varepsilon}_R e^c \gamma_c \gamma_5 \gamma_{ab} \psi_L\right) + \text{ h.c.} \tag{85}$$

Putting together the gamma matrices and using the duality relation $\gamma_5 \gamma_{abc} = -i\varepsilon_{abcd}\gamma^d$, we obtain

合并伽马矩阵并利用对偶关系 $\gamma_5 \gamma_{abc} = -i\varepsilon_{abcd}\gamma^d$ ，我们得到

(86)

$$\begin{aligned}
\delta_g \mathcal{L}_{\mathcal{M}_\psi} &= M_P^3 \frac{|g|^2}{4} e^a e^b e^c \varepsilon_{abcd} \left(\overline{\psi}_L \gamma^d \varepsilon_R + \overline{\psi}_R \gamma^d \varepsilon_L\right) \\
&= -M_P^4 \frac{|g|^2}{2} e^a e^b e^c \varepsilon_{abcd} \frac{\overline{\varepsilon}_R \gamma^d \psi_L + \overline{\varepsilon}_L \gamma^d \psi_R}{2M_P} \\
&= -M_P^4 \frac{|g|^2}{2} e^a e^b e^c \varepsilon_{abcd} \delta e^d \\
&= -|g|^2 \frac{M_P^4}{8} \delta \left(e^a e^b e^c e^d \varepsilon_{abcd}\right).
\end{aligned}$$

We further recall from Appendix ”Vierbein and Cartan’s Formalism” that $e^a e^b e^c e^d \varepsilon_{abcd} = +4! d^4 xe$ and then realize that we need to add a single term of order $|g|^2$ to the Lagrangian to cancel (86):

我们再从附录“Vierbein 与嘉当形式体系”回顾 $e^a e^b e^c e^d \varepsilon_{abcd} = +4! d^4 xe$ ，可知需要给拉格朗日量添加一个单个的 $|g|^2$ 阶项来抵消 (86) 式:

$$3\int d^4 x e M_P^4 |g|^2 = -M_P^2 \int d^4 x e \Lambda. \tag{87}$$

This is a cosmological constant term. Notice that there is no choice of the sign of this cosmological constant

这就是宇宙常数项。注意该宇宙常数的符号是固定的，没有选择余地

$$\Lambda = -3M_P^2 |g|^2 = -\frac{3}{\ell^2} < 0. \tag{88}$$

This agrees with the discussion following from the supersymmetry algebra.

这与从超对称代数出发得到的讨论一致。

It is also extremely important to note that the variation of (87) does not generate terms of order g^3 and that therefore supersymmetry closes at order g^2 .

另外还需特别注意:(87) 式的变分不会产生 g^3 阶项，因此超对称在 g^2 阶闭合。

The only variation left is the vierbein variation in the gravitino mass term together with the already mentioned first term in (83). Using

仅剩的变分是引力微子质量项中的 vierbein 变分，加上之前提到的 (83) 式中的第一项，利用

$$4\left(\overline{\psi}_R \wedge \gamma_{ab}\psi_R\right) \wedge \left(\bar{\varepsilon}\gamma^b\psi\right) = 3\left(\overline{\psi}_R\gamma_{ab}\varepsilon_R\right) \wedge \left(\overline{\psi} \wedge \gamma^b\psi\right) + \left(\overline{\psi}_R\varepsilon_R\right) \wedge \left(\overline{\psi} \wedge \gamma_a\psi\right),$$

(89)

which one can derive from the Fierz identities (212)-(216), one obtains for all remaining uncancelled variations

它可由 Fierz 恒等式 (212)-(216) 导出，对于所有剩余未抵消的变分，我们得到

$$\delta\mathcal{L} =$$

$$\frac{M_P}{2}\left(De^a - \frac{1}{4M_P^2}\overline{\psi} \wedge \gamma^a\psi\right) \wedge \Big[ig^* M_P e^b \wedge \left(\bar{\varepsilon}_L\gamma_{ba}\psi_L - \frac{1}{2}\overline{\psi}_L\gamma_b\gamma_a\varepsilon_L\right)$$

$$-igM_P e^b \wedge \left(\bar{\varepsilon}_R\gamma_{ba}\psi_R - \frac{1}{2}\overline{\psi}_R\gamma_b\gamma_a\varepsilon_R\right)$$

$$+\varepsilon_{abcd}\left(-\frac{1}{6}\overline{D\psi}\gamma^{bcd}\varepsilon + M_P\delta\omega^{bc} \wedge e^d\right)\Big], \tag{90}$$

where the last term proportional to ε_{abcd} is the same as in the case without cosmological constant.

其中正比于 ε_{abcd} 的最后一项与无宇宙常数的情形相同。

This completes the proof of the invariance of the action in any of the formalisms described above. In the second-order formalism, this variation vanishes because of the torsion constraint. In the first-order formalism, we deduce from this variation the expression for $\delta\omega^{ab}$ that makes it vanish. Finally, in the 1.5-order formalism, the equations of motion for the spin connection do not change, and hence, once again, the full Lagrangian is invariant under supersymmetry.

至此我们完成了上述任意形式体系下作用量不变性的证明。在二阶形式体系中，由于挠率约束，该变分为零。在一阶形式体系中，我们可以从该变分得到令其为零的 $\delta\omega^{ab}$ 的表达式。最后，在 1.5 阶形式体系中，自旋联络的运动方程不发生改变，因此整个拉格朗日量仍然在超对称变换下保持不变。

Bringing the mass term (84) to the standard form without differential forms, the final Lagrangian is therefore the following:

将质量项 (84) 整理为不含微分形式的标准形式，可得最终拉格朗日量如下:

$$\mathcal{L} = \frac{M_P^2}{2} eR - \frac{e}{2}\overline{\psi}_{\mu R}\gamma^{\mu\upsilon\rho}D_\upsilon\psi_{\rho L} - \frac{e}{2}\overline{\psi}_{\mu L}\gamma^{\mu\upsilon\rho}D_\upsilon\psi_{\rho R}$$

$$-eM_P\frac{g}{2}\overline{\psi}_{\mu R}\gamma^{\mu\upsilon}\psi_{\upsilon R} - eM_P\frac{g^*}{2}\overline{\psi}_{\mu L}\gamma^{\mu\upsilon}\psi_{\upsilon L} \tag{91}$$

$$+3eM_P^4|g|^2,$$

where we should remember that in the second-order formalism, $\omega^{ab} = \omega^{ab}(e, \psi)$. The final supersymmetry transformations are

其中我们需要记住，在二阶形式体系中，$\omega^{ab} = \omega^{ab}(e, \psi)$。最终的超对称变换为

$$\delta e_\mu^a = \frac{1}{2M_P}\bar{\varepsilon}_L\gamma^a\psi_{\mu R} + \text{ h.c. }, \tag{92}$$

$$\delta\psi_{\mu L} = M_P D_\mu\varepsilon_L - \frac{g}{2}M_P^2\gamma_\mu\varepsilon_R. \tag{93}$$

Now that we completed the construction of a supersymmetric action for super-gravity with a (negative) cosmological constant, we can make some comments.

我们完成带 (负) 宇宙常数的超引力超对称作用量构造后，可以做一些讨论。

First of all, we have seen from the construction that we have performed only minimal modifications. After shifting the supersymmetry transformation of the gravitino field, we introduced the smallest set of terms needed to cancel $O(g)$ and $O(g^2)$ terms in the supersymmetry variations. As already pointed out above, supersymmetry closes at order $|g|^2$. There is no need to introduce any term of order g^3 or more.

首先，我们从构造过程中可以看到，我们仅做了最小程度的修改。平移引力微子场的超对称变换后，我们引入了最小集合的项，刚好抵消超对称变换中的 $O(g)$ 和 $O(g^2)$ 项。正如上文已经指出的，超对称在 $|g|^2$ 阶闭合，不需要引入任何 g^3 阶或更高阶的项。

All the modifications can be summarized in three main pieces:

所有修改可以总结为三个主要部分:

- A shift in the fermionic supersymmetry rules at $\mathcal{O}(g)$

- 费米子超对称规则在 $\mathcal{O}(g)$ 处的平移

- A mass-like term of $\mathcal{O}(g)$ for the fermions

- 费米子的 $\mathcal{O}(g)$ 阶类质量项

- A potential term at order $\mathcal{O}(g^2)$

- $\mathcal{O}\left(g^2\right)$ 阶的势项

Although these modifications have been forced by the presence of the cosmological constant in a pure gravity theory, one finds the pattern outlined above also in all gauged supergravity theories for models with extended supersymmetries. In fact, in extended supergravities, the appearance of non-Abelian gauge groups is tied to the presence of a non-trivial scalar potential, which may act as an effective cosmological constant. The result is that the gauging procedure introduces the same three main modifications listed above, where the mass-like term for the fermions in the general case with scalar fields becomes a Yukawa-like coupling and the cosmological constant term becomes a scalar potential.

尽管这些修改是纯引力理论中存在宇宙学常数所必须的，人们发现上述结构同样存在于所有具有扩展超对称的定域规范超引力理论中。实际上，在扩展超引力中，非阿贝尔规范群的出现与非平凡标量势的存在相关，而该标量势可以充当等效宇宙学常数。结果就是，定域规范引入了和上文列出的完全相同的三项主要修改，其中在包含标量场的一般情况下，费米子的类质量项变为汤川耦合，宇宙学常数项变为标量势。

It can also be seen that this scalar potential (in this case a pure cosmological constant) can be expressed as the square of the shifts of the supersymmetry variations of the fermionic fields:

还可以发现，该标量势 (在本文情况中就是纯宇宙学常数) 可以表示为费米子场超对称变换平移量的平方:

$$V\bar{\varepsilon}_L\gamma^a\varepsilon_R = -3M_P^4\big|_g|g|^2\bar{\varepsilon}_L\gamma^a\varepsilon_R = -\frac{3}{2}\delta_g\overline{\psi}_{\mu R}\gamma^a\delta_g\psi_L^\mu. \tag{94}$$

Note the minus sign in front of the squared gravitino shifts. This identity is called the supersymmetric Ward identity [28].

注意引力微子平移量平方前的负号。这个恒等式被称为超对称沃德恒等式 [28]。

Matter Couplings in Supergravity

超引力中的物质耦合

 Once the construction of the pure supergravity theory has been completed, we can try and analyze what needs to be done to couple matter multiplets in a consistent way. This is what we are going to discuss in this section. In our presentation, we will limit as much as possible the details of the derivations and focus instead on the new features of matter couplings in supergravity as compared to what is already required by rigid supersymmetry. In fact, already relaxing the requirement of studying renormalizable interactions generalizes quite a lot the possible couplings of globally supersymmetric theories, without the need of resorting to (super)gravity. We would therefore like in the following to pinpoint the signatures that are unique to the supersymmetrization of the gravitational interaction.

本节重印自 [1] ©2021 Springer-Verlag GmbH Germany，属于施普林格自然旗下。已获授权重印。版权所有。在完成纯超引力理论的构造后，我们可以分析如何以自洽的方式耦合物质多重态。这正是我们本节要讨论的内容。在阐述中，我们会尽可能简化推导细节，转而聚焦超引力中物质耦合相比刚性超对称要求的新特征。事实上，仅放宽可重整相互作用的研究要求，就已经能大幅推广整体超对称理论的可能耦合形式，无需借助 (超) 引力。因此我们下文会明确指出引力相互作用超对称化独有的特征。

The discussion on the matter couplings clearly depends heavily on what kind of matter we allow in these couplings and if we allow more than two-derivative terms. More general Lagrangians could be obtained by introducing additional matter multiplets, such as tensor multiplets. The tensor fields in such tensor multiplets, however, can in general be dualized to either massless scalar or massive vector fields so that the theory will be eventually of the standard form we will write down. On the other hand, these dualities are often non-perturbative or may require complicated field redefinitions. It may therefore be interesting to study these models directly with tensor fields, also because such tensor fields naturally arise from string theory compactifications, but we will not discuss this here.

关于物质耦合的讨论显然高度依赖我们允许哪些类型的物质参与耦合，以及是否允许高于二阶导数的项。引入额外的物质多重态 (例如张量多重态) 可以得到更一般的拉格朗日量。不过这类张量多重态中的张量场通常可以对偶化为无质量标量场或有质量矢量场，因此理论最终会呈现为我们将要写出的标准形式。但另一方面，这些对偶往往是非微扰的，或是需要复杂的场重新定义。因此直接用张量场研究这类模型也有其意义——这类张量场自然出现在弦理论紧化中，但我们在此不做讨论。

Other generalizations may include higher degree form fields, higher spin fields, and/or higher derivative terms. While for each of these generalizations one can work out the construction that successfully leads to supersymmetric theories, we will stick to the simplest minimal theory coupled to vector and chiral multiplets, which are enough to provide rich theories with matter and gauge couplings.

其他推广方向可包括高阶形式场、高自旋场和/或高阶导数项。尽管对每一类推广都可以构造出自洽的超对称理论，我们仍会坚持最简单的极小理论，即耦合矢量多重态和手征多重态，二者已经足以给出包含物质耦合与规范耦合的丰富理论。

In the following, we denote the complex scalars of n_C chiral multiplets by ϕ^m $(m, n, \ldots = 1, \ldots, n_C)$ and their complex conjugates by $\phi^{\bar{m}}$ and use χ_L^m and $\chi_R^{\bar{m}}$ for the chiral projections of their fermionic superpartners. The component fields of n_V vector multiplets are the gaugini, λ^I , and the vector fields, $A_\mu^I(I, J, \ldots = 1, \ldots, n_V)$

下文中，我们将 n_C 个手征多重态的复标量记为 ϕ^m $(m, n, \ldots = 1, \ldots, n_C)$，其复共轭记为 $\phi^{\bar{m}}$，并使用 χ_L^m 和 $\chi_R^{\bar{m}}$ 表示它们费米超对称伙伴的手征投影。n_V 个矢量多重态的分量场是戈希尼 λ^I 和矢量场 $A_\mu^I(I, J, \ldots = 1, \ldots, n_V)$

General $\mathcal{N} = 1$ globally supersymmetric theories of chiral and vector multiplets are completely specified by the following data:

一般的 $\mathcal{N} = 1$ 维整体超对称手征与矢量多重态理论完全由以下数据确定:

- The numbers, n_C and n_V , of the chiral and vector multiplets

- 手征多重态和矢量多重态的数目 n_C 和 n_V

- The Kähler potential $K\left(\phi^m, \phi^{\bar{m}}\right)$ that determines the geometry of the scalar manifold, $\mathcal{M}_{\text{scalar}}$

- 决定标量流形几何 $\mathcal{M}_{\text{scalar}}$ 的凯勒势 $K\left(\phi^m, \phi^{\bar{m}}\right)$

- The holomorphic superpotential $W\left(\phi^m\right)$ that encodes the self-interactions of the chiral multiplets

- 编码手征多重态自相互作用的全纯超势 $W\left(\phi^m\right)$

- The holomorphic gauge kinetic function $f_{IJ}\left(\phi^m\right)$ related to the kinetic terms of the vector multiplets

- 与矢量多重态动能项相关的全纯规范动力学函数 $f_{IJ}\left(\phi^m\right)$

- The action of the gauge group on $\mathcal{M}_{\text{scalar}}$, as specified by the holomorphic Killing vectors, $\xi_I^m\left(\phi^n\right)$, and the corresponding Killing prepotentials, $\mathcal{P}_I\left(\phi^m, \phi^{\bar{m}}\right)$

- 规范群在 $\mathcal{M}_{\text{scalar}}$ 上的作用，由全纯基灵矢量 $\xi_I^m\left(\phi^n\right)$ 和对应的基灵预势 $\mathcal{P}_I\left(\phi^m, \phi^{\bar{m}}\right)$ 给出

- The real Fayet-Iliopoulos terms, η_I , which might be non-zero for Abelian gauge group factors

- 实费耶特-伊利亚普洛斯项 η_I ，该项对阿贝尔规范群因子可以非零

When we couple such a theory to supergravity, making it locally supersymmetric, there will be additional couplings of the matter multiplets to the supergravity multiplet, but also new and modified couplings among the fields of the matter multiplets themselves [29-31]. All these additional or modified couplings are still completely specified by the abovementioned data that already specified a theory in global supersymmetry. As we will now explain, they will appear in the Lagrangian with inverse powers of M_P .

当我们将这类理论耦合到超引力，使其成为局域超对称理论时，除了物质多重态会与超引力多重态产生额外耦合外，物质多重态自身的场之间也会出现新的、经修正的耦合 [29-31]。所有这些额外或经修正的耦合，仍然完全由已经指定了整体超对称理论的上述数据确定。正如我们接下来会说明的，它们会在拉格朗日量中以 M_P 的负幂次形式出现。

Coupling Chiral Multiplets to Supergravity

手征多重态与超引力的耦合

Let us start with the modifications that are necessary in order to make a globally supersymmetric Wess-Zumino model also invariant under local supersymmetry. The globally supersymmetric Lagrangian in Minkowski space is

我们首先介绍必要的修改，使整体超对称韦斯-祖米诺模型也满足局域超对称性下的不变性。闵氏空间中的整体超对称拉格朗日量为

$$\mathcal{L}_{WZ} = -g_{m\bar{n}} \left[\left(\partial_\mu \phi^m\right) \left(\partial^\mu \phi^{\bar{n}}\right) + \overline{\chi}_L^m \mathcal{D} \chi_R^{\bar{n}} + \overline{\chi}_R^{\bar{n}} \mathcal{D} \chi_L^m \right]$$

$$- \left(\mathcal{D}_m \partial_n W\right) \overline{\chi}_L^m \chi_L^n - \left(\mathcal{D}_{\bar{m}} \partial_{\bar{n}} W^*\right) \overline{\chi}_{\bar{R}}^{\bar{m}} \chi_R^{\bar{n}} \tag{95}$$

$$-g^{m\bar{n}} \left(\partial_m W\right) \left(\partial_{\bar{n}} W^*\right) + O\left(\chi^4\right),$$

where $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K$ is the Kähler metric. The derivative $\mathcal{D}$ is covariant with respect to arbitrary holomorphic scalar field reparameterizations and hence contains the Christoffel symbols, Γ_{mn}^p , on the scalar manifold, e.g.,

> 其中 $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K$ 为凯勒度量。导数 $\mathcal{D}$ 对任意全纯标量场重参数化是协变的，因此标量流形上包含克里斯托费尔符号 Γ_{mn}^p ，例如

$$\mathcal{D}_\mu \chi_L^m \equiv \partial_\mu \chi_L^m + \left(\partial_\mu \phi^n\right) \Gamma_{nl}^m \chi_L^l. \tag{96}$$

If we now allow for a spacetime-dependent supersymmetry parameter, $\varepsilon = \varepsilon(x)$, the derivative in the kinetic terms of the fermions χ^m will produce new terms when it acts on $\varepsilon(x)$ coming from the supersymmetry transformations of the chiral fermions

> 如果我们现在考虑依赖时空的超对称参数 $\varepsilon = \varepsilon(x)$，费米子动能项中的导数 χ^m 作用在手征费米子超对称变换给出的 $\varepsilon(x)$ 上时，会产生新项

$$\delta \chi_L^m = \frac{1}{2} \oint \phi^m \varepsilon_R - \frac{1}{2} g^{m\bar{n}} \left(\partial_{\bar{n}} W^*\right) \varepsilon_L + O(\chi\chi\varepsilon), \tag{97}$$

$$\delta \chi_R^{\bar{m}} = \frac{1}{2} \partial\!\!\!/ \phi^{\bar{m}} \varepsilon_L - \frac{1}{2} g^{\bar{m}n} \left(\partial_n W\right) \varepsilon_R + O(\chi\chi\varepsilon). \tag{98}$$

The result is an uncancelled variation of the form

> 结果会得到如下形式的未抵消变分

$$\delta \mathcal{L}_{\mathrm{WZ}} = \bar{J}_R^\mu \partial_\mu \varepsilon_R + \bar{J}_L^\mu \partial_\mu \varepsilon_L, \tag{99}$$

where the supercurrents are

> 其中超流为

$$\bar{J}_L^\mu = -g_{m\bar{n}} \overline{\chi}_L^m \gamma^\mu \partial \phi^{\bar{n}} + \overline{\chi}_R^{\bar{n}} \gamma^\mu \partial_{\bar{n}} W^*, \; J_R^\mu = \left(J_L^\mu\right)^c. \tag{100}$$

As we have already shown in section ”Promoting Supersymmetry to a Local Symmetry” for the special case of a free Wess-Zumino model, the cancellation of these terms is achieved by adding the Noether couplings to the gravitino

正如我们已经在“将超对称推广为局域对称性”一节中对自由韦斯-祖米诺模型特例所展示的，这些项可以通过向引力微子添加诺特耦合来抵消

$$\mathcal{L}_{\text{Noether}} = -\frac{1}{M_P}\left[\bar{J}_R^{\mu}\psi_{\mu R} + \bar{J}_L^{\mu}\psi_{\mu L}\right]. \tag{101}$$

Using $\delta\psi_\mu = M_P\partial_\mu\varepsilon$, one then finds that everything cancels modulo terms that come from the variation of the supercurrents themselves:

利用 $\delta\psi_\mu = M_P\partial_\mu\varepsilon$ ，可以得到，除了超流自身变分产生的项外，所有项都相互抵消:

$$\delta\left(\mathcal{L}_{WZ} + \mathcal{L}_{\text{Noether}}\right) = -\frac{1}{M_P}\left[\left(\delta\bar{J}_R^{\mu}\right)\psi_{\mu R} + \text{ h.c. }\right] \tag{102}$$

These terms are of the form

这些项形式如下

$$\delta\left(\mathcal{L}_{WZ} + \mathcal{L}_{\text{Noether}}\right) = -\delta g^{\mu\upsilon}T_{\mu\upsilon} + Z_1 + Z_2. \tag{103}$$

Just as discussed in section "Promoting Supersymmetry to a Local Symmetry," $T_{\mu\nu}$ is the energy momentum tensor of $\mathcal{L}_{WZ}$, and the new field $g_{\mu\nu}$ is identified with the spacetime metric, signalling the necessity for a coupling to gravity. The minimal coupling to a dynamical metric is achieved by covariantizing everything with respect to general spacetime coordinate and local Lorentz transformations and by adding the pure supergravity Lagrangian. The metric variation of this covariantized Lagrangian then precisely cancels the first term in (103), and the theory would be supersymmetric if there weren't also the two additional terms Z_1 and Z_2 in Eq. (103) that we have neglected so far. As we will now show, these two terms are actually quite important, as they lead to additional M_P^{-2} -suppressed interactions between the fields of the chiral multiplets themselves that have some far-reaching consequences.

正如我们在“将超对称推广为局域对称性”一节中讨论的，$T_{\mu\nu}$ 是 $\mathcal{L}_{WZ}$ 的能量动量张量，新场 $g_{\mu\nu}$ 被等同于时空度规，这说明必须耦合引力。通过对一般时空坐标变换和局域洛伦兹变换做协成化，再加上纯超引力拉格朗日量，就得到了动力学度规的最小耦合。该协成化拉格朗日量的度规变分恰好抵消了 (103) 式的第一项，如果到目前为止我们忽略的式 (103) 中两个额外项 Z_1 和 Z_2 不存在的话，该理论就满足超对称性。我们接下来会说明，这两项实际上非常重要，它们会导致手征多重态各场之间额外的、被 M_P^{-2} 压低的相互作用，带来一系列影响深远的结论。

In order to make this more precise, let us first state what Z_1 and Z_2 are

为了表述更精确，我们首先说明 Z_1 和 Z_2 分别是什么

$$Z_1 = -\frac{e}{2M_P}g_{m\bar{n}}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\gamma_5\varepsilon\left(\partial_\upsilon\phi^m\right)\left(\partial_\rho\phi^{\bar{n}}\right), \tag{104}$$

$$Z_2 = \frac{e}{M_P}\left[\overline{\psi}_{\mu L}\gamma^{\mu\upsilon}\varepsilon_L\left(\partial_\upsilon W^*\right) + \overline{\psi}_{\mu R}\gamma^{\mu\upsilon}\varepsilon_R\left(\partial_\upsilon W\right)\right]. \tag{105}$$

The first term Z_1 comes from the variations of the form $\delta\chi_L^m \sim \frac{1}{2}\partial\!\!\!/\phi^m\varepsilon_R$ in J_L^μ and its conjugate, which give rise to terms with three antisymmetrized gamma matrices as well as terms with one gamma matrix. The latter are part of the energy momentum tensor terms in (103) (because $\delta g_{\mu\nu}$ involves only one gamma matrix), whereas the terms with three antisymmetrized gamma matrices are precisely given by Z_1 . The first term of Z_2 is due to the variations $\delta\chi_L^m \sim -\frac{1}{2}g^{m\bar{n}}\left(\partial_{\bar{n}}W^*\right)\varepsilon_L$ in the first term in (100) and due to the variation $\delta\chi_R^{\bar{m}} \sim \frac{1}{2}\partial\phi^{\bar{m}}\varepsilon_L$ in the second term in (100). The second term in Z_2 arises from the analogous variations of J_R^μ .

第一项 Z_1 来自 J_L^μ 中 $\delta\chi_L^m \sim \frac{1}{2}\partial\!\!\!/\phi^m\varepsilon_R$ 形式的变分及其共轭变分，产生包含三个反对称伽马矩阵的项和包含一个伽马矩阵的项。后者属于 (103) 式中能量动量张量项 (因为 $\delta g_{\mu\nu}$ 仅包含一个伽马矩阵)，而包含三个反对称伽马矩阵的项恰好就是 Z_1 给出的。Z_2 的第一项来自 (100) 式第一项中 $\delta\chi_L^m \sim -\frac{1}{2}g^{m\bar{n}}\left(\partial_{\bar{n}}W^*\right)\varepsilon_L$ 的变分，以及 (100) 式第二项中 $\delta\chi_R^{\bar{m}} \sim \frac{1}{2}\partial\phi^{\bar{m}}\varepsilon_L$ 的变分。Z_2 的第二项来自 J_R^μ 的类似变分。

We will now see that the cancellation of Z_1 and Z_2 requires the introduction of new terms with important consequences.

我们接下来会看到，抵消 Z_1 和 Z_2 需要引入新项，这些新项会带来重要结论。

The Kähler-Covariant Derivative

凯勒协变导数

In order to cancel Z_1 , we first rewrite it by using the relation between the metric of the scalar manifold and the Kähler potential:

为了消去 Z_1 ，我们首先利用标量流形度量与凯勒势之间的关系重写该项:

$$Z_1 \sim \overline{\psi}_\mu\gamma^{\mu\nu\rho}\gamma_5\varepsilon\left(\partial_\nu\phi^m\right)\left(\partial_\rho\phi^{\bar{n}}\right)\partial_m\partial_{\bar{n}}K$$

(106)

$$= \overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\gamma_5\varepsilon\frac{1}{2}\left(\partial_\rho\phi^{\bar{n}}\partial_\upsilon\partial_{\bar{n}}K - \partial_\rho\phi^m\partial_\upsilon\partial_m K\right).$$

Using the last expression, we then integrate by parts the spacetime derivative that acts on the Kähler potential. This produces in particular terms where the derivative acts on ε and terms where it acts on $\overline{\psi}_\mu$. The former term is

利用上述表达式，我们对作用在凯勒势上的时空导数做分部积分。这会得到两类项，一类导数作用于 ε ，另一类导数作用于 $\overline{\psi}_\mu$ ，其中第一类项为

$$\frac{e}{2M_P}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\gamma_5\left(D_\upsilon\varepsilon\right)\frac{1}{2}\left(\partial_\rho\phi^{\bar{n}}\partial_{\bar{n}}K - \partial_\rho\phi^m\partial_m K\right). \tag{107}$$

We now repeat our old trick and simply add the negative of this term (times a factor 1/2) to the Lagrangian, but with $D_\upsilon\varepsilon$ replaced by ψ_υ ,

现在我们沿用之前的方法: 直接将该项的负值 (乘以因子 1/2)) 加入拉格朗日量，只需将 $D_\upsilon\varepsilon$ 替换为 ψ_υ ,

$$\mathcal{L}_{\text{Kähler cov}} = -\frac{e}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\left(\frac{i}{2M_P^2}Q_\upsilon(\phi)\gamma_5\right)\psi_\rho, \tag{108}$$

where Q_υ is a composite vector field,

其中 Q_υ 是复合矢量场,

$$Q_\upsilon(\phi) \equiv \frac{i}{2}\left[(\partial_{\bar{n}}K)\,\partial_\upsilon\phi^{\bar{n}} - (\partial_m K)\,\partial_\upsilon\phi^m\right]. \tag{109}$$

Varying the two gravitini in this expression would then precisely cancel (107).

对该表达式中的两个引力微变分就恰好能消去式 (107)。

The cancellation of the remaining term in Z_1 , where the derivative acts on $\overline{\psi}_\mu$, will be discussed later (See the text below eq. (115)).

Z_1 中剩余项 (导数作用于 $\overline{\psi}_\mu$ 的项) 的消去我们会在后续讨论 (见式 (115) 下方的内容)。

The new interaction term $\mathcal{L}_{\text{Kähler cov}}$, however, now poses another problem: as one easily verifies, it is not invariant under Kähler transformations $K \to K+h+h^*$. $\mathcal{L}_{\text{Kähler cov}}$ would thus seem to single out a particular Kähler potential, even though a specific Kähler potential is not an intrinsic geometrical object on a Kähler manifold. In general, the Kähler potential is in fact only locally defined and requires Kähler transformations on the overlaps of local coordinate patches. So if the Lagrangian was not Kähler-invariant, the physics would in general also be different for different coordinate patches of the scalar manifold.

但新的相互作用项 $\mathcal{L}_{\text{Kähler cov}}$ 带来了另一个问题: 不难验证，它在凯勒变换 $K \to K+h+h^*$ 下不不变。因此 $\mathcal{L}_{\text{Kähler cov}}$ 似乎会预先选定一个特定的凯勒势，然而凯勒势本身并不是凯勒流形上的内禀几何对象。一般而言，凯勒势实际上仅在局域定义，在局域坐标邻域的交叠区需要进行凯勒变换。因此如果拉格朗日量不满足凯勒不变性，标量流形不同坐标邻域对应的物理规律一般也会不同。

To understand the resolution of this problem, we observe that the term $\mathcal{L}_{\text{Kähler}}$ cov can be absorbed into the Rarita-Schwinger action by modifying the covariant derivative with a new term,

为了解决这个问题，我们发现可以通过对协变导数添加新项，将协变项 $\mathcal{L}_{\text{Kähler}}$ 吸收进拉里塔-施温格尔作用量,

$$\mathcal{L}_{\text{RS}} + \mathcal{L}_{\text{Kähler cov}} = -\frac{e}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\mathcal{D}_\upsilon(\omega,Q)\,\psi_\rho, \tag{110}$$

where

其中

$$\mathcal{D}_{[\upsilon}(\omega, Q)\,\psi_{\rho]} \equiv D_{[\upsilon}(\omega)\,\psi_{\rho]} + \frac{i}{2M_P^2} Q_{[\upsilon}\gamma_5\psi_{\rho]}, \tag{111}$$

with D_υ being the Lorentz-covariant derivative. To understand the significance of this modification, one notes that Q_μ transforms under Kähler transformations like a U(1) connection:

> D_υ 是洛伦兹协变导数。为了理解这一修改的意义，请注意 Q_μ 在凯勒变换下的变换性质与 U(1) 联络一致:

$$Q_\mu \to Q_\mu + \partial_\mu \operatorname{Im}(h)\,. \tag{112}$$

More precisely, Q_μ is a composite U (1) connection, i.e., it is not an elementary vector field, but rather a function of the scalar fields and their derivatives.

> 更准确地说，Q_μ 是复合 U (1) 联络，即它不是基本矢量场，而是标量场及其导数的函数。

We now see that we can render the Lagrangian-invariant if we require that Kähler transformations, $K \to K + h + h^*$, be accompanied by chiral rotations of the gravitino:

> 我们现在发现，如果要求凯勒变换 $K \to K + h + h^*$ 伴随引力微子的手征旋转，就能让拉格朗日量满足不变性:

$$\psi_\mu \to \exp\left[-\frac{i}{2M_P^2} \operatorname{Im}(h(\phi))\,\gamma_5\right]\psi_\mu. \tag{113}$$

Indeed, the derivative (111) then transforms covariantly,

> 不难看出，此时式 (111) 的导数会协变变换,

$$\mathcal{D}_{[\mu]}\psi_{\rho]} \to \exp\left[-\frac{i}{2M_P^2} \operatorname{Im}(h(\phi))\,\gamma_5\right]\mathcal{D}_{[\mu]}\psi_{\rho]}, \tag{114}$$

and the combination (110) is Kähler-(and obviously also locally Lorentz-) invariant. These geometric arguments thus suggest that, in supergravity, Kähler transformations on the scalar manifold also act on the gravitino as a chiral U(1) symmetry, with Q_μ being the corresponding (composite) U (1) connection. If this is to make sense, this non-trivial action of Kähler transformations on the gravitini should also be compatible with supersymmetry. As we will now show, this requirement will lead to further interesting differences with respect to global supersymmetry and provides further consistency checks.

> 而式 (110) 的组合满足凯勒不变性 (显然也满足局域洛伦兹不变性)。这些几何论证表明，在超引力中，标量流形上的凯勒变换也会作为手征 U(1) 对称性作用在引力微子上，其中 Q_μ 是对应的 (复合) U (1) 联络。若要这个结论自洽，凯勒变换对引力微子的非平凡作用必须还能与超对称相容。我们接下来会说明，这个要求会带来与整体超对称进一步的有趣差异，也给出了更多自洽性检验。

First we note that if the gravitino transforms under Kähler transformations, the consistency with the supersymmetry transformation law $\delta\psi_\mu \sim M_P D_\mu \varepsilon$ also requires that ε transforms under Kähler transformations,

首先我们注意到，如果引力微子在凯勒变换下发生变换，那么与超对称变换律 $\delta\psi_\mu \sim M_P D_\mu \varepsilon$ 的相容性也要求 ε 在凯勒变换下发生变换,

$$\varepsilon \to \exp\left[-\frac{i}{2M_P^2}\,\mathrm{Im}\,(h\,(\phi))\,\gamma_5\right]\varepsilon, \tag{115}$$

and that its derivative (as it appears in $\delta\psi_\mu$) should also be covariantized (This is indeed confirmed by computing the gravitino variations of $\mathcal{L}_{RS}$ with the new Kähler-covariant transformation law, $\delta\psi_\mu \sim M_P \mathcal{D}_\mu \varepsilon$, which leads to a new term that precisely cancels the remaining uncancelled part of Z_1 (i.e., the part of Z_1 with a derivative acting on $\overline{\psi}_\mu$).),

并且其导数 (正如它出现在 $\delta\psi_\mu$ 中的形式) 也应当协变化 (这一点可以通过计算 $\mathcal{L}_{RS}$ 在新的凯勒协变变换定律 $\delta\psi_\mu \sim M_P \mathcal{D}_\mu \varepsilon$ 下的引力微子变分得到证实，计算会产生一个新项，恰好抵消了 Z_1 中剩余未抵消的部分，即 Z_1 中作用于 $\overline{\psi}_\mu$ 的含导数部分),

$$\mathcal{D}_\mu\,(\omega, Q)\,\varepsilon \equiv D_\mu\,(\omega)\,\varepsilon + \frac{i}{2M_P^2} Q_\mu \gamma_5 \varepsilon. \tag{116}$$

This in turn implies, because of $\delta\chi_L^m = \frac{1}{2}\partial\!\!\!/ \phi^m \varepsilon_R + \ldots$, that also the chiral fermions transform under Kähler transformations,

反过来，由于 $\delta\chi_L^m = \frac{1}{2}\partial\!\!\!/ \phi^m \varepsilon_R + \ldots$ ，这也意味着手征费米子会在凯勒变换下发生变换,

$$\chi^m \to \exp\left[+\frac{i}{2M_P^2}\,\mathrm{Im}\,(h\,(\phi))\,\gamma_5\right]\chi^m, \tag{117}$$

and that their derivatives have to be Lorentz-, $\mathcal{M}_{\text{scalar}}$ reparameterization-, and Kähler-covariant, e.g., (For the sake of simplicity, we do not introduce a new symbol for the Kähler-covariantized derivative and still call it $\mathcal{D}_\mu$.)

并且它们的导数必须同时满足洛伦兹协变、$\mathcal{M}_{\text{scalar}}$ 重参数化协变与凯勒协变，例如，(为简化起见，我们不为凯勒协变化后的导数引入新符号，仍将其记作 $\mathcal{D}_\mu$ 。)

$$\mathcal{D}_\mu \chi_L^m \equiv D_\mu \chi_L^m + \left(\partial_\mu \phi^n\right)\Gamma^m_{nl}\chi_L^l - \frac{i}{2M_P^2} Q_\mu \chi_L^m. \tag{118}$$

Note that there is a different sign in (117) (and hence also in (118)) compared to the corresponding terms of the gravitino or the supersymmetry transformation parameter (cf. (113) and (115) as well as (111) and (116)). This sign difference arises because one has to move the γ_5 matrix in (115) through one gamma matrix in the supersymmetry transformation $\delta\chi_L^m = \frac{1}{2}\partial\phi^m \varepsilon_R + \ldots$

请注意，与引力微子或超对称变换参数的对应项 (参见 (113)、(115) 以及 (111)、(116)) 相比，(117)(因此也包括 (118)) 中的符号存在差异。该符号差异的来源是，必须将 (115) 中的 γ_5 矩阵移动，穿过超对称变换 $\delta\chi_L^m = \frac{1}{2}\partial\phi^m \varepsilon_R + \ldots$ 中的一个伽马矩阵

Although we will discuss gauge multiplets later, we already mention here that $\delta\lambda^I \sim \frac{1}{4}\gamma^{\mu\upsilon}\mathcal{F}^I_{\mu\upsilon}\varepsilon + \ldots$ implies that also the gaugini transform non-trivially under Kähler transformations (with the same sign as ψ_μ and ε)

尽管我们会在后面讨论规范多重态，但我们在此提前说明: $\delta\lambda^I \sim \frac{1}{4}\gamma^{\mu\upsilon}\mathcal{F}^I_{\mu\upsilon}\varepsilon + \ldots$ 表明，微子也会在凯勒变换下发生非平凡变换 (符号与 ψ_μ 和 ε 一致)

$$\delta\lambda^I \to \exp\left[-\frac{i}{2M_P^2}\operatorname{Im}\left(h\left(\phi\right)\right)\gamma_5\right]\lambda^I \tag{119}$$

and that likewise all their derivatives have to be properly covariantized with respect to Kähler transformations (again with the same sign as for ψ_μ and ε).

同理，它们的所有导数也必须针对凯勒变换做适当的协变化 (符号同样与 ψ_μ 和 ε 一致)。

To conclude, all fermion fields and not just the gravitino are charged with respect to a composite chiral U(1) symmetry that is related to Kähler transformations and that is not present in the global case. It should be emphasized that in the limit of global supersymmetry, $M_P \to \infty$, these chiral rotations become trivial, as is signalled by the inverse powers of M_P . This is consistent with the rigid supersymmetry Lagrangian (95), where this chiral composite U(1) is not encountered.

综上，所有费米子场 (而非仅引力微子) 都相对于与凯勒变换相关的复合手征 U(1) 对称性带荷，该对称性在全局超对称情形中不存在。需要强调的是，在全局超对称极限 $M_P \to \infty$ 下，这些手征旋转会变为平凡变换，这由 M_P 的逆幂次表明。这与刚性超对称拉格朗日量 (95) 一致，其中不存在该复合手征 U(1)。

Interestingly, the above non-trivial transformations of the fermions under Kähler transformations also imply that the superpotential and its derivatives have to transform as we will show in section ”Additional Bare Superpotential Terms.” The result is that

有趣的是，上述费米子在凯勒变换下的非平凡变换还意味着超势及其导数必须按我们将在“额外裸超势项”一节中说明的方式变换，结论是

$$W \to \exp\left[-\frac{1}{M_P^2}h\left(\phi^m\right)\right]W\left(\phi^m\right) \tag{120}$$

and its derivatives have to be Kähler-covariantized as follows:

且它的导数必须按下式做凯勒协变化:

$$\partial_n W \to e^{\frac{K}{2M_P^2}}\mathcal{D}_n W \equiv e^{\frac{K}{2M_P^2}}\left[\partial_n + \frac{(\partial_n K)}{M_P^2}\right]W. \tag{121}$$

To summarize: The cancellation of Z_1 by adding $\mathcal{L}_{\text{Kähler cov}}$ gives rise to the interpretation that the fermions and the superpotential should transform non-trivially under Kähler transformations. In order to ensure this, all derivatives of the fermions and the superpotential have to be Kähler-covariantized, and the superpotential terms have to be dressed with an exponential of the Kähler potential. One can show that all these modifications are indeed also necessary for the cancellation of various other variations we have not discussed here in detail. In general, we define the Kähler-covariant derivatives in field space as

总结: 通过添加 $\mathcal{L}_{\text{Kähler cov}}$ 抵消 Z_1 的过程可以得到这样的结论: 费米子和超势应当在凯勒变换下发生非平凡变换。为保证这一点，费米子和超势的所有导数都必须做凯勒协变化，且超势项必须乘以凯勒势的指数。可以证明，所有这些修改对于抵消我们此处未详细讨论的其他各类变分而言确实也是必要的。一般来说，我们将场空间中的凯勒协变导数定义为

$$D_m \Phi = \left(\partial_m + \frac{p}{M_P^2} \partial_m K\right) \Phi,$$

(122)

$$\bar{D}_{\bar{m}} \Phi = \left(\bar{\partial}_{\bar{m}} - \frac{p}{M_P^2} \bar{\partial}_{\bar{m}} K\right) \Phi,$$

where p is the Kähler "charge" of the field Φ .

其中 p 是场 Φ 的凯勒“荷”。

Additional Bare Superpotential Terms

额外裸超势项

In global supersymmetry, all superpotential terms always appear with at least one derivative with respect to the scalar fields. As we saw in the previous subsection, the coupling to supergravity (in particular the cancellation of the term Z_1) requires a Kähler covariantization of these derivatives of W , which then introduces "bare" W -terms inside these Kähler-covariant derivatives, i.e., W -terms that are not differentiated with respect to any scalar field. In this subsection, we show that there are additional "bare" superpotential terms in the Lagrangian and the supersymmetry transformation laws. Their necessity follows from the cancellation of the term Z_2 to which we now turn.

在全局超对称中，所有超势项都至少带有一个对标量场的导数。正如我们在上一小节看到的，耦合到超引力 (尤其是抵消项 Z_1) 要求对 W 的这些导数做凯勒协变化，这会在凯勒协变导数内部引入“裸” W 项，即不对标量场求导的 W 项。本小节我们将证明，拉格朗日量和超对称变换律中存在额外的“裸”超势项。这些项的必要性来自项 Z_2 的抵消，我们现在就来讨论这一点。

In order to cancel the Z_2 -term,

为了抵消 Z_2 项,

$$Z_2 \equiv \frac{e}{M_P}\left[\overline{\psi}_{\mu L}\gamma^{\mu\upsilon}\varepsilon_L\left(\partial_\upsilon W^*\right) + \overline{\psi}_{\mu R}\gamma^{\mu\upsilon}\varepsilon_R\left(\partial_\upsilon W\right)\right], \tag{123}$$

we proceed as we did for Z_1 and first perform an integration by parts. This will then give again terms with a derivative acting on the supersymmetry parameter ε and terms where the derivative acts on the gravitini $\overline{\psi}_\mu$

. To cancel the former, we then again add to the Lagrangian a term where the derivatives of ε are replaced by gravitini or, more precisely,

我们采用和处理 Z_1 时相同的方法，首先做分部积分。这会再次得到两类项: 一类是导数作用在超对称参数 ε 上的项，另一类是导数作用在引力微子 $\overline{\psi}_\mu$ 上的项。为了抵消前一类项，我们再次向拉格朗日量中添加一项，将 ε 的导数替换为引力微子，更准确地说:

$$\frac{e}{2M_P^2}\left[W^*\overline{\psi}_{\mu L}\gamma^{\mu\upsilon}\psi_{\upsilon L}+W\overline{\psi}_{\mu R}\gamma^{\mu\upsilon}\psi_{\upsilon R}\right]. \tag{124}$$

This term is an obvious mass-like term for the gravitino, and therefore, following the rules we have learned in the case of pure supergravity in the presence of a cosmological constant, we have to further modify the variation of the gravitino field by adding a new term of the form

该项是引力微子一个显然的质量类项，因此，根据我们在带宇宙学常数的纯超引力情形中学到的规则，我们必须进一步修改引力微子场的变分，添加一个如下形式的新项

$$\delta_{\text{ new}}\,\psi_{\mu L}\sim\frac{1}{2M_P}W\gamma_\mu\varepsilon_R. \tag{125}$$

This new variation applied to the Rarita-Schwinger action also gives the term required to cancel the second piece coming from the partial integration of Z_2 , namely, the term with the derivative acting on the gravitino.

将这个新变分应用到拉里塔-施温格作用量，也会得到抵消 Z_2 分部积分产生的第二部分所需的项，也就是导数作用在引力微子上的那项。

Before proceeding further, let us come back to the Kähler covariantization of the superpotential terms and prove (120). Subjecting (125) to Kähler transformations tells us that the left-hand side transforms as

在进一步讨论之前，我们回到超势项的凯勒协变性，证明式 (120)。对 (125) 做凯勒变换可知，左侧的变换为

$$\exp\left[-\frac{1}{4M_P^2}\left(h\left(\phi\right)-h^*\left(\phi\right)\right)\right], \tag{126}$$

while the epsilon parameter on the right-hand side transforms with the opposite sign due to the opposite chirality:

而由于手性相反，右侧的 epsilon 参数的变换符号相反:

$$\exp\left[+\frac{1}{4M_P^2}\left(h\left(\phi\right)-h^*\left(\phi\right)\right)\right]. \tag{127}$$

At this point, it is obvious that in order for the Kähler transformation to be compatible with supersymmetry, we need to transform also the superpotential, as we already mentioned earlier. The superpotential, on the other hand, is a holomorphic function by construction and hence can transform only with a holomorphic factor,

至此很明显，为了让凯勒变换和超对称相容，我们还必须变换超势，这一点我们之前已经提过。另一方面，超势按构造是全纯函数，因此只能带一个全纯因子变换，

$$W \to \exp\left[-\frac{\alpha}{M_P^2}\left(h\left(\phi\right)\right)\right] W, \tag{128}$$

where α is a real constant. In order to get the same rotation on the left- and on the right-hand side of (125), we still need something that transforms under Kähler transformations with the exponential of $h + h^*$, like the exponential of the Kähler potential itself, $\mathrm{e}^{\beta K/M_P^2}$. The right coefficients follow then by equating the two sides:

其中 α 是一个实常数。为了让式 (125) 的左右两侧得到相同的转动，我们仍然需要一个在凯勒变换下按 $h + h^*$ 的指数变换的对象，就像凯勒势本身的指数 $\mathrm{e}^{\beta K/M_P^2}$ 那样。然后令两侧相等即可得到正确系数:

$$-\frac{1}{4M_P^2}\left(h\left(\phi\right) - h^*\left(\phi\right)\right) = +\frac{1}{4M_P^2}\left(h\left(\phi\right) - h^*\left(\phi\right)\right) - \frac{\alpha}{M_P^2}\left(h\left(\phi\right)\right) + \frac{\beta}{M_P^2}\left(h\left(\phi\right) + h^*\left(\phi\right)\right).$$
(129)

This fixes $\alpha = 1$ and $\beta = 1/2$ and tells us that we have to replace the superpotential with the combination

这就确定了 $\alpha = 1$ 和 $\beta = 1/2$，说明我们必须将超势替换为如下组合

$$\mathrm{e}^{K/\left(2M_P^2\right)} W \tag{130}$$

and that indeed W transforms under Kähler transformations as in (120).

且确实 W 会按照式 (120) 做凯勒变换。

Coming back to the check of supersymmetry invariance, we now see that the new transformation law for the gravitino (125) applied to the new bilinear term (124) gives a new variation of the form $|W|^2\overline{\psi}\gamma\varepsilon$. Not too surprisingly, this can then finally be cancelled by adding a new contribution $\sim -e|W|^2$ to the scalar potential and varying the vierbein determinant e. This is the generalization of the procedure derived in section ”Adding a Cosmological Constant” for the case of a constant superpotential, i.e., for pure supergravity with a cosmological constant.

回到超对称不变性的验证，我们现在可以看到，将新的引力微子变换律 (125) 应用到新的双线性项 (124)，会得到形式为 $|W|^2\overline{\psi}\gamma\varepsilon$ 的新变分。不难发现，这个变分最终可以通过对标量势添加一个新贡献 $\sim -e|W|^2$ 并变分 Vierbein 行列式 e 来抵消。这是对“添加宇宙学常数”一节中得到的 procedure 的推广，原 procedure 适用于常数超势的情形，也就是带宇宙学常数的纯超引力。

Although it may be hard to believe, it turns out that, after proper Kähler covari-antizations, the above modifications are sufficient to ensure also the cancellations of all the other variations we have not considered explicitly here.

或许难以置信，但结果表明，经过适当的凯勒协变化后，上述修改足以保证我们在此没有明确讨论的所有其他变分都能被抵消。

The end result is the Lagrangian

最终得到的拉格朗日量为

$$
\begin{aligned}
e^{-1}\mathcal{L} = & \frac{M_P^2}{2}R(e,\omega(e)) - \frac{1}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\mathcal{D}_\upsilon(\omega(e),Q)\psi_\rho \\
& -g_{m\bar{n}}\left[(\partial_\mu\phi^m)(\partial^\mu\phi^{\bar{n}}) + \overline{\chi}_L^m \mathcal{D}\chi_R^{\bar{n}} + \overline{\chi}_R^{\bar{n}}\mathcal{D}\chi_L^m\right] \\
& -\left\{e^{K/2M_P^2}(\mathcal{D}_m\mathcal{D}_n W)\overline{\chi}_L^m\chi_L^n + \text{ h.c. }\right\} \qquad (131) \\
& +\frac{1}{M_P}\left\{g_{m\bar{n}}\overline{\psi}_{\mu L}\gamma^\nu\gamma^\mu\chi_L^m(\partial_\nu\phi^{\bar{n}}) + \overline{\psi}_{\mu R}\gamma^\mu\chi_L^m e^{K/2M_P^2}\mathcal{D}_m W + \text{ h.c. }\right\} \\
& +\frac{1}{2M_P^2}\left\{e^{K/2M_P^2}W\overline{\psi}_{\mu R}\gamma^{\mu\upsilon}\psi_{\upsilon R} + \text{ h.c}\right\} - V(\phi^m,\phi^{\bar{n}}),
\end{aligned}
$$

with the scalar potential given by the sum of two contributions

其中标量势由两项贡献相加给出

$$
V = e^{K/M_P^2}\left[g^{m\bar{n}}(\mathcal{D}_m W)(\mathcal{D}_{\bar{n}}W^*) - \frac{3|W|^2}{M_P^2}\right], \qquad (132)
$$

where the first term is the Kähler covariantization of the F-terms from global super-symmetry and the second is a genuine contribution from gravitational couplings, in the sense that it is a variation of the vierbein determinant that leads to a cancellation of the $|W|^2$ terms mentioned after (130). In the Lagrangian (131), the first line is the Kähler covariantization of the pure supergravity action. The second and third lines correspond to the Kähler- and spacetime-covariant Wess-Zumino action (without the potential). Note that now

其中第一项是整体超对称中 F 项的凯勒协变化，第二项是引力耦合的真贡献: 它是 vierbein 行列式的变分，可抵消 (130) 式后提到的 $|W|^2$ 项。在拉格朗日量 (131) 中，第一行是纯超引力作用量的凯勒协变化。第二行和第三行对应凯勒与时空协变的韦斯-祖米诺作用量 (不含势)。请注意

$$
\mathcal{D}_m\mathcal{D}_n W = \left(\partial_m + \frac{\partial_m K}{M_P^2}\right)\left[\left(\partial_n + \frac{\partial_n K}{M_P^2}\right)W\right] - \Gamma_{mn}{}^p\left(\partial_p + \frac{\partial_p K}{M_P^2}\right)W.
$$

(133)

The fourth line is the Kähler- and spacetime-covariant Noether coupling of the supercurrents to the gravitino, $\mathcal{L}_{\text{Noether}} = -\frac{1}{M_P}\left[\bar{J}_R^\mu\psi_{\mu R} + \bar{J}_L^\mu\psi_{\mu L}\right]$ (with the fermions moved into a different order). The fifth line, finally, contains the W -dependent extra terms as well as the (Kähler-covariantized) scalar potential of the Wess-Zumino model. The supersymmetry transformation rules, up to three-fermion terms, are

第四行是超流与引力微子的凯勒、时空协变诺特耦合，$\mathcal{L}_{\text{Noether}} = -\frac{1}{M_P}\left[J_R^{\mu}\psi_{\mu R} + J_L^{\mu}\psi_{\mu L}\right]$ (费米子已调整为不同顺序)。第五行最终包含依赖 W 的额外项，以及韦斯-祖米诺模型的 (凯勒协变化的) 标量势。超对称变换规则在三费米子项范围内可写为

$$\delta e_{\mu}^{a} = \frac{1}{2M_P}\bar{\varepsilon}\gamma^{a}\psi_{\mu}$$

$$\delta\psi_{\mu L} = M_P\mathcal{D}_{\mu}\left(\omega\left(e\right), Q_v\right)\varepsilon_L + \frac{1}{2M_P}e^{K/2M_P^2}W\gamma_{\mu}\varepsilon_R, \tag{134}$$

$$\delta\phi^{m} = \bar{\varepsilon}_L\chi_L^{m}$$

$$\delta\chi_L^{m} = \frac{1}{2}\partial\!\!\!/\phi^{m}\varepsilon_R - \frac{1}{2}g^{m\bar{n}}e^{K/2M_P^2}\left(\mathcal{D}_{\bar{n}}W^{*}\right)\varepsilon_L.$$

Obviously, in the $M_P \to \infty$ limit, these equations reduce to the globally supersymmetric theory. One also notices that truncating out the chiral multiplets and keeping a constant superpotential $\mathrm{e}^{\frac{K}{2M_P^2}}W = -gM_P^3$ give back the pure supergravity Lagrangian with cosmological constant, Eq. (91).

显然，在 $M_P \to \infty$ 极限下，这些方程退化为整体超对称理论。还可以发现，截断手征多重态并保留常数超势 $\mathrm{e}^{\frac{K}{2M_P^2}}W = -gM_P^3$ 后，我们就回到带宇宙学常数的纯超引力拉格朗日量，即式 (91)。

Note further that in supergravity, the Kähler potential and the superpotential are no longer independent, as one can shift terms back and forth via Kähler transformations. In fact, as long as W is not equal to zero, one can even make the superpotential equal to M_P^3 by performing a Kähler transformation with $h\left(\phi\right) = M_P^2\log\left(W/M_P^3\right)$ (cf. Eq. (120)). More generally, instead of using the two functions K and W, one can express the entire Lagrangian in terms of the function

还需注意，在超引力中凯勒势和超势不再独立，因为可以通过凯勒变换来回移动项。实际上，只要 W 不等于零，我们甚至可以通过用 $h\left(\phi\right) = M_P^2\log\left(W/M_P^3\right)$ 做凯勒变换，让超势变为 M_P^3 (参见式 (120))。更一般地，我们无需使用 K 和 W 这两个函数，而是可以用函数将整个拉格朗日量写为

$$\mathcal{G} = K + M_P^2\log\frac{|W|^2}{M_P^6}, \tag{135}$$

which is manifestly Kähler-invariant. For instance, the part of the scalar potential coming from the superpotential becomes

它具有明显的凯勒不变性。例如，标量势中来自超势的部分变为

$$V = e^{\mathcal{G}/M_P^2}\left(M_P^2g^{m\bar{n}}\mathcal{G}_m\mathcal{G}_{\bar{n}} - 3M_P^4\right). \tag{136}$$

Note, however, that by doing so, one cannot recover the $W = 0$ case, which has to be discussed separately, hence the usefulness of leaving explicit both K and W in our approach.

但请注意，这样做无法得到 $W = 0$ 的情况，该情况必须单独讨论，这也是我们的方法中将 K 和 W 都保留为显式的好处。

Inclusion of Vector Multiplets

矢量多重态的引入

The inclusion of vector multiplets requires the following changes:

引入矢量多重态需要进行以下修改:

1. All terms that were already present in global supersymmetry are also present in supergravity, but they all have to be made spacetime- and Kähler-covariant.

1. 全局超对称中原本就存在的所有项在超引力中也同样存在，但所有项都必须改写为时空协变且凯勒协变的形式。

2. A new Noether coupling of the vector multiplet supercurrent

2. 矢量多重态超流的新诺特耦合

$$\bar{J}^{\mu}_{VM} \equiv e\overline{\lambda}^{J}\left[-\frac{1}{4}(\mathrm{Re}\, f_{IJ})\mathcal{F}^{I}_{\upsilon\rho}\gamma^{\mu}\gamma^{\upsilon\rho} - \frac{i}{2}\mathcal{P}_{J}\gamma^{\mu}\gamma_{5}\right] \tag{137}$$

to the gravitino has to be introduced:

必须引入引力微子的耦合项:

$$\mathcal{L}'_{\text{Noether}} = -\frac{1}{M_P}\bar{J}^{\mu}_{VM}\psi_{\mu} \tag{138}$$

in order to cancel terms of the form

目的是抵消形如

$$\delta\mathcal{L} = \bar{J}^{\mu}_{VM}\partial_{\mu}\varepsilon \tag{139}$$

that arise due to the derivative in $-\frac{1}{2}e(\mathrm{Re}\, f_{IJ})\overline{\lambda}^{I}\hat{\not{\partial}}\lambda^{J}$ when it acts on the ε in $\delta\lambda$. In the above equations, $\mathcal{F}^{I}_{\mu\nu}$ denotes the usual gauge-covariant field strengths, and a hat on a derivative denotes a gauge-covariant derivative.

当 $-\frac{1}{2}e(\mathrm{Re}\, f_{IJ})\overline{\lambda}^{I}\hat{\not{\partial}}\lambda^{J}$ 中的导数作用于 $\delta\lambda$ 中的 ε 时产生的项。上述方程中，$\mathcal{F}^{I}_{\mu\nu}$ 表示常规的规范协变场强，导数上的帽子表示规范协变导数。

3. The composite Kähler connection Q_{μ} receives an additional contribution proportional to $A^{I}_{\mu}\mathcal{P}_{I}$ for each of the gauged isometries:

3. 对每个规范等距变换，复合凯勒联络 Q_{μ} 会获得一项正比于 $A^{I}_{\mu}\mathcal{P}_{I}$ 的额外贡献:

$$Q_\mu = Q_\mu\left(\phi^m, \phi^{\bar n}, A^I_\mu\right) = \frac{i}{2}\left(\left(\partial_{\bar n} K\right)\partial_\mu \phi^{\bar n} - \left(\partial_m K\right)\partial_\mu \phi^m\right) + A^I_\mu \mathcal{P}_I \tag{140}$$

$$= \frac{i}{2}\left(\left(\partial_{\bar n} K\right)\hat\partial_\mu \phi^{\bar n} - \left(\partial_m K\right)\hat\partial_\mu \phi^m\right) + A^I_\mu \,\mathrm{Im}\left(r_I\right), \tag{141}$$

where the last equality follows from the form of the prepotentials (we will see more on this in section "More on D-Terms"). This additional term is needed, e.g., in order to cancel a variation proportional to $\mathcal{F}^J_{\mu\nu}\mathcal{P}_J\bar\varepsilon\gamma^{\mu\upsilon\rho}\gamma_5\psi_\rho$ that occurs in the variation $-\frac{1}{M_P}\delta\left(J^\mu_{VM}\right)\psi_\mu$ and is not of the form $-\delta g^{\mu\upsilon}T_{\mu\upsilon}$.

其中最后一个等式由预备势的形式导出 (我们会在“D 项补充”一节中详细说明)。这项额外项是必需的，例如用于抵消变分 $-\frac{1}{M_P}\delta\left(J^\mu_{VM}\right)\psi_\mu$ 中出现的正比于 $\mathcal{F}^J_{\mu\nu}\mathcal{P}_J\bar\varepsilon\gamma^{\mu\upsilon\rho}\gamma_5\psi_\rho$ 、且不属于 $-\delta g^{\mu\upsilon}T_{\mu\upsilon}$ 形式的变分项。

It should be noted that this additional contribution to Q_μ has another important consequence. Namely, if one shifts the Killing prepotential $\mathcal{P}_I$ of an Abelian factor by a Fayet-Iliopoulos constant, $\mathcal{P}_I \to \mathcal{P}_I + \eta_I$, one introduces new chiral gauge interactions for all fermions, including, e.g., the gravitino,

需要注意的是，这项对 Q_μ 的额外贡献还有另一个重要结论: 即如果将阿贝尔因子的基林预备势 $\mathcal{P}_I$ 偏移一个费耶特-伊里奥普洛斯常数 $\mathcal{P}_I \to \mathcal{P}_I + \eta_I$ ，就会引入所有费米子的新手征规范相互作用，其中包括引力微子,

$$\mathcal{D}_\mu\psi_\upsilon \to \mathcal{D}_\mu\psi_\upsilon + \frac{i}{2M_P^2}A^I_\mu\eta_I\gamma_5\psi_\upsilon \tag{142}$$

which can easily lead to quantum anomalies [32]. Thus, the introduction of Fayet-Iliopoulos constants in $\mathcal{N} = 1$ supergravity requires some care. We will actually see later on that in supergravity, the Fayet-Iliopoulos terms are related to the non-invariance of the superpotential under gauge transformations.

这很容易引发量子反常 [32]。因此，在 $\mathcal{N} = 1$ 超引力中引入费耶特-伊里奥普洛斯常数需要谨慎。我们后续会看到，在超引力中，费耶特-伊里奥普洛斯项与超势在规范变换下的非不变性相关。

Ignoring four-fermion terms, the end result of all these modifications is the following general matter-coupled Lagrangian (When the gauge kinetic function is not gauge-invariant, so-called generalized Chern-Simons terms of the form $A^I \wedge A^J \wedge dA^K$ and $A^I \wedge A^J \wedge A^K \wedge A^L$ may be possible. Their form, however, is the same as in global supersymmetry [33].):s

忽略四费米子项，所有这些修改后的最终结果是以下通用的物质耦合拉格朗日量 (当规范动力学函数不满足规范不变性时，可能存在形如 $A^I \wedge A^J \wedge dA^K$ 和 $A^I \wedge A^J \wedge A^K \wedge A^L$ 的所谓广义陈-西蒙斯项。然而它们的形式与全局超对称中的形式一致 [33]。):

$$e^{-1}\mathcal{L} = \frac{M_P^2}{2}R\left(e, \omega\left(e\right)\right) - \frac{1}{2}\overline{\psi}_\mu\gamma^{\mu\upsilon\rho}\mathcal{D}_\upsilon\left(\omega\left(e\right), Q\right)\psi_\rho$$

$$-g_{m\bar n}\left[\left(\hat\partial_\mu\phi^m\right)\left(\hat\partial^\mu\phi^{\bar n}\right) + \overline{\chi}^m_L\widehat{\mathcal{D}}\chi^{\bar n}_R + \overline{\chi}^{\bar n}_R\widehat{\mathcal{D}}\chi^m_L\right]$$

$$+\left(\mathrm{Re}\, f_{IJ}\right)\left[-\frac{1}{4}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{\mu\nu J} - \frac{1}{2}\overline{\lambda}^I\widehat{\mathcal{D}}\lambda^J\right]$$

$$+\frac{1}{8}\left(\mathrm{Im}\, f_{IJ}\right)\left[\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} - 2i\widehat{\mathcal{D}}_\mu\left(e\overline{\lambda}^I\gamma_5\gamma^\mu\lambda^J\right)\right]$$

$$+\left\{-\frac{1}{4}f_{IJ,m}\mathcal{F}^I_{\mu\nu}\overline{\chi}^m_L\gamma^{\mu\nu}\lambda^J_L + \frac{i}{2}D^I f_{IJ,m}\overline{\chi}^m_L\lambda^J\right.$$

$$+\frac{1}{4}e^{K/2M_P^2}\left(\mathcal{D}_m W\right)g^{m\bar{n}}f^*_{IJ,\bar{n}}\overline{\lambda}^I_R\lambda^J_R$$

$$\left.-e^{K/2M_P^2}\left(\mathcal{D}_m\mathcal{D}_n W\right)\overline{\chi}^m_L\chi^n_L - 2\xi^{\bar{n}}_I g_{m\bar{n}}\overline{\lambda}^I\chi^m_L + \text{ h.c.}\right\}$$

$$+\frac{1}{4M_P}\left(\mathrm{Re}\, f_{IJ}\right)\overline{\psi}_\mu\gamma^{\nu\rho}\gamma^\mu\lambda^J\mathcal{F}^I_{\nu\rho}$$

$$+\left\{\frac{1}{M_P}g_{m\bar{n}}\overline{\psi}_{\mu L}\gamma^\nu\gamma^\mu\chi^m_L\left(\hat{\partial}_\nu\phi^{\bar{n}}\right) + \text{ h.c.}\right\}$$

$$+\frac{1}{M_P}\left\{\overline{\psi}_{\mu R}\gamma^\mu\left[\frac{i}{2}\lambda^I_L\mathcal{P}_I + \chi^m_L e^{K/2M_P^2}\mathcal{D}_m W\right] + \text{ h.c.}\right\}$$

$$+\frac{1}{2M_P^2}\left\{e^{K/2M_P^2}W\overline{\psi}_{\mu R}\gamma^{\mu\upsilon}\psi_{\upsilon R} + \text{ h.c}\right\} - V\left(\phi^m, \phi^{\bar{n}}\right), \tag{143}$$

with the scalar potential

其中标量势为

$$V = e^{K/M_P^2}\left[g^{m\bar{n}}\left(\mathcal{D}_m W\right)\left(\mathcal{D}_{\bar{n}}\bar{W}\right) - \frac{3|W|^2}{M_P^2}\right] + \frac{1}{2}\left(\mathrm{Re}\, f_{IJ}\right)D^I D^J. \tag{144}$$

The supersymmetry transformation rules, up to three-fermion terms, are

超对称变换规则 (忽略三阶费米子项以上) 为

$$\delta e^a_\mu = \frac{1}{2M_P}\bar{\varepsilon}\gamma^a\psi_\mu$$

$$\delta\psi_{\mu L} = M_P\mathcal{D}_\mu\left(\omega\left(e\right), Q\right)\varepsilon_L + \frac{1}{2M_P}e^{K/2M_P^2}W\gamma_\mu\varepsilon_R,$$

$$\delta\phi^m = \bar{\varepsilon}_L\chi^m_L$$

$$\delta\chi^m_L = \frac{1}{2}\partial\!\!\!/\phi^m\varepsilon_R - \frac{1}{2}g^{m\bar{n}}e^{K/\left(2M_P^2\right)}\left(\mathcal{D}_{\bar{n}}W^*\right)\varepsilon_L,$$

$$\delta A^I_\mu = -\frac{1}{2}\bar{\varepsilon}\gamma_\mu\lambda^I$$

$$\delta\lambda^I = \frac{1}{4}\gamma^{\mu\upsilon}\mathcal{F}^I_{\mu\upsilon}\varepsilon + \frac{i}{2}\gamma_5 D^I\varepsilon \tag{145}$$

It is again easy to see that, in the global limit, $M_P \to \infty$, the above equations reduce to the globally supersymmetric theory.

容易看出，在全局极限 $M_P \to \infty$ 下，上述方程退化为全局超对称理论。

For completeness, we display the full (i.e., local Lorentz-, scalar coordinate-, Kähler-, and gauge-covariant) derivative of λ^I and χ^m :

为完整起见，我们给出 λ^I 和 χ^m 的全协变导数 (即同时满足局部洛伦兹协变、标量坐标协变、凯勒协变和规范协变):

$$\widehat{\mathcal{D}}_\mu \chi_L^m = D_\mu \chi_L^m + \left(\hat{\partial}_\mu \phi^n\right) \Gamma_{nl}^m \chi_L^l - A_\mu^I \left(\partial_n \xi_I^m\right) \chi_L^n - \frac{i}{2M_P^2} Q_\mu \chi_L^m,$$

$$\widehat{\mathcal{D}}_\mu \lambda^I = D_\mu \lambda^I + A_\mu^J f_{JK}^I \lambda^K + \frac{i}{2M_P^2} Q_\mu \gamma_5 \lambda^I, \tag{146}$$

where, as usual, D_μ denotes the Lorentz-covariant derivative. The full covariant derivatives of ψ_μ and ε are just as for λ^I , except for the gauge covariantization term $A_\mu^J f_{JK}^I \lambda^K$, which is absent for these fermions (hence, we can omit the hat on their derivatives).

和通常情况一样, 此处 D_μ 表示洛伦兹协变导数。ψ_μ 和 ε 的完整协变导数与 λ^I 的形式完全一致, 唯一区别是规范协变项 $A_\mu^J f_{JK}^I \lambda^K$ ，这些费米子不存在该项 (因此我们可以对它们的导数省略尖帽标记)。

More details on the action and the four-fermion terms can be found in [33,34].

关于该作用量和四费米子项的更多细节可参见文献 [33,34]。

More on D-Terms

更多关于 D 项的内容

Although the D-terms and the D-term potential take the same form as in global supersymmetry, local supersymmetry does have some interesting implications also for the D-terms. To understand this, we recall that the general matter-coupled supergravity Lagrangian is invariant under Kähler transformations that act at the same time on the Kähler potential, the superpotential, and the fermions. As in global supersymmetry, a gauge transformation therefore does not necessarily have to leave the Kähler potential invariant, but may in general transform it with a Kähler transformation,

尽管 D 项和 D 项势的形式与全局超对称中的形式相同，但局域超对称对 D 项也确实存在一些有趣的推论。为理解这一点，我们回顾: 通用物质耦合超引力拉格朗日量在同时作用于凯勒势、超势和费米子的凯勒变换下保持不变。因此和全局超对称的情况一样，规范变换不一定非要保持凯勒势不变，一般来说它可以通过凯勒变换来变换凯勒势,

$$\delta_{\text{gauge}} K \equiv \xi_I^m \partial_m K + \xi_I^{\bar{m}} \partial_{\bar{m}} K = r_I + r_I^*. \tag{147}$$

However, in supergravity theories, this requires a non-trivial action also on the superpotential (if $W \neq 0$
)

但在超引力理论中，这要求对超势也存在非平凡作用 (如果 $W \neq 0$)

$$\delta_{\text{gauge}} W \equiv \xi_I^m \partial_m W = -\frac{r_I}{M_P^2} W, \tag{148}$$

so that the combination $\mathcal{G}$ in (135) remains invariant:

因此 (135) 中的组合 $\mathcal{G}$ 保持不变:

$$\delta_{\text{gauge}} \mathcal{G} = \xi_I^m \partial_m \mathcal{G} + \xi_I^{\bar{m}} \partial_{\bar{m}} \mathcal{G} = 0. \tag{149}$$

For all the points in field space where $W \neq 0$, we can then also rewrite r_I as

对于场空间中所有满足 $W \neq 0$ 的点，我们还可以将 r_I 改写为

$$r_I = -M_P^2 \xi_I^m \frac{\partial_m W}{W}, \tag{150}$$

so that the Killing prepotentials can be also expressed in terms of the gauge-invariant quantity $\mathcal{G}$:

因此基林预势也可以用规范不变量 $\mathcal{G}$ 表示为:

$$\mathcal{P}_I = i\xi_I^m \partial_m K - i r_I = i\xi_I^m \left[\partial_m K + \frac{M_P^2 \partial_m W}{W} \right] = i\xi_I^m \frac{M_P^2 \mathcal{D}_m W}{W} = i\xi_I^m \partial_m \mathcal{G}.$$

(151)

The D-terms are thus

因此 D 项为

$$D^I = i(\mathrm{Re}\, f)^{-1IJ} \xi_J^m \partial_m \mathcal{G}, \tag{152}$$

and the total scalar potential with F-terms and D-terms can be written in a very compact and suggestive form

包含 F 项和 D 项的总标量势可以写成非常简洁且具有启发性的形式

$$V = e^{\mathcal{G}/M_P^2} \left[h^{m\bar{n}} \mathcal{G}_m \mathcal{G}_{\bar{n}} - 3 \right] M_P^2, \tag{153}$$

where

其中

$$h^{m\bar{n}} \equiv g^{m\bar{n}} + \frac{e^{-\mathcal{G}/M_P^2}}{2} (\mathrm{Re}\, f)^{-1IJ} \xi_I^m \xi_J^{\bar{n}}, \tag{154}$$

so that the new metric contains the Kähler metric giving the F-term potential (136) and the additional term coming from the D-term potential.

因此新度规包含给出 F 项势 (136) 的凯勒度规，以及来自 D 项势的附加项。

We can now comment on some of the differences between global and local supersymmetry, which may lead to relevant physical differences.

现在我们来讨论全局超对称和局域超对称之间的一些差异，这些差异可能会带来显著的物理区别。

First of all, we see that in supergravity, D-terms and F-terms are not independent of one another; rather, the D-terms are (for $W \neq 0$) a particular combination of the F-terms (152).

首先我们可以看到，在超引力中，D 项和 F 项并非相互独立；恰恰相反，(对于 $W \neq 0$)D 项是 F 项 (152) 的一种特殊组合。

The next important difference concerns the Fayet-Iliopoulos constants. Just as in global supersymmetry, the gauge transformation (147) of the Kähler potential fixes r_I only up to an additive imaginary constant, $i\eta_I$, and hence $\mathcal{P}_I$ up to an additive real constant η_I . However, consistency of the $\mathcal{P}_I$ moment maps with the gauge symmetry requires the equivariance condition

下一个重要差异与法耶特-伊利亚普洛斯 (Fayet-Iliopoulos，简称 FI) 常数有关。和全局超对称的情况一样，凯勒势的规范变换 (147) 仅确定 r_I 到一个附加虚常数 $i\eta_I$ ，因此 $\mathcal{P}_I$ 也仅确定到一个附加实常数 η_I 。但 $\mathcal{P}_I$ 矩量与规范对称性的一致性要求等变性条件

$$\{\mathcal{P}_I, \mathcal{P}_J\} = \frac{1}{2}\left(\delta_I \mathcal{P}_J - \delta_J \mathcal{P}_I\right) = f_{IJ}{}^K \mathcal{P}_K \tag{155}$$

and this restricts the possible values of the η_I constants except for $\mathrm{U}(1)$ factors. The difference to global supersymmetry now is that the superpotential W also transforms under gauge transformations as in (148) so that a shift of r_I by an additive constant $i\eta_I$ implies that W transforms with an additional phase factor under the corresponding $\mathrm{U}(1)$ transformation. In other words, changing η_I changes the $\mathrm{U}(1)$ charge of W . Note that such $\mathrm{U}(1)$ transformations due to FI constants may even occur when the Kähler potential is invariant under this $\mathrm{U}(1)$ factor, because, according to (147), this only implies $r_I(\phi) = i\eta_I$.

这限制了 η_I 常数除 $\mathrm{U}(1)$ 因子外的所有可能取值。现在它和全局超对称的区别在于，超势 W 也会像 (148) 那样在规范变换下变换，因此 r_I 被附加常数 $i\eta_I$ 平移后，意味着 W 会在对应的 $\mathrm{U}(1)$ 变换下附加一个相位因子变换。换句话说，改变 η_I 就会改变 W 的 $\mathrm{U}(1)$ 荷。需要注意的是，这种由 FI 常数导致的 $\mathrm{U}(1)$ 变换甚至可以在凯勒势对该 $\mathrm{U}(1)$ 因子不变的情况下发生，因为根据 (147)，这仅能推出 $r_I(\phi) = i\eta_I$ 。

Another important effect of a FI constant is that it leads to a chiral U(1) transformation of the fermions, as follows from their non-trivial transformations under Kähler transformations described in section ”The Kähler-Covariant Derivative.” As explained around (142), this may then easily lead to anomalous gauge couplings and requires some care.

FI 常数的另一个重要效应是，它会导致费米子的手征 U(1) 变换，这可以从“凯勒协变导数”章节中描述的费米子在凯勒变换下的非平凡变换推导得出。正如 (142) 附近的说明所述，这很容易导致反常规范耦合，因此需要格外注意。

Appendix: Conventions, Spinors, and Useful Relations

附录: 约定、旋量与有用关系式

General Relativity and Spacetime Conventions

广义相对论与时空约定

We use x^μ $(\mu, \upsilon, \ldots = 0, 1, 2, 3)$ to denote local 4D spacetime coordinates and ∂_μ as the corresponding coordinate basis vectors of the tangent spaces.

我们用 x^μ $(\mu, \upsilon, \ldots = 0, 1, 2, 3)$ 表示局域四维时空坐标，用 ∂_μ 表示切空间对应的坐标基矢。

The metric tensor has components $g_{\mu\nu}$ and signature $(- + ++)$, and the Christoffel symbols, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$, always refer to the usual torsion-free Levi-Civita connection,

度量张量的分量为 $g_{\mu\nu}$ ，号差为 $(- + ++)$ ，克里斯托费尔符号 $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ 始终对应常规无挠列维-奇维塔联络,

$$\Gamma^\rho_{\mu\upsilon} = \Gamma^\rho_{\mu\upsilon}(g) = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\upsilon\sigma} + \partial_\upsilon g_{\mu\sigma} - \partial_\sigma g_{\mu\upsilon}\right). \tag{156}$$

The curvature tensor is defined as

曲率张量定义为

$$R_{\rho\sigma}{}^{\mu}{}_{\upsilon} \equiv 2\partial_{[\rho}\Gamma^{\mu}_{\sigma]\upsilon} + 2\Gamma^{\mu}_{\tau[\rho}\Gamma^{\tau}_{\sigma]\upsilon}, \tag{157}$$

where here and throughout the text we use symmetrization () and antisymmetriza-tion [] with weight one, i.e., $(\mu\upsilon) = 1/2\,(\mu\upsilon + \upsilon\mu)$, $[\mu\upsilon] = 1/2\,(\mu\upsilon - \upsilon\mu)$, etc.

本文全文均使用权重为 1 的对称化 () 和反对称化 [], 即如 $(\mu\upsilon) = 1/2\,(\mu\upsilon + \upsilon\mu)$, $[\mu\upsilon] = 1/2\,(\mu\upsilon - \upsilon\mu)$ 这类形式。

The Ricci tensor, the Ricci scalar, and the Einstein-Hilbert action are given by

里奇张量、里奇标量与爱因斯坦-希尔伯特作用量分别为

$$R_{\mu\nu} \equiv R_{\rho\mu}{}^{\rho}{}_{\nu} \tag{158}$$

$$R \equiv R_{\rho\mu}{}^{\rho}{}_{\upsilon} g^{\mu\upsilon} = R_{\mu\upsilon} g^{\mu\upsilon}, \tag{159}$$

$$S_{EH} = \frac{M_P^2}{2} \int d^4x\sqrt{-g}R = \frac{1}{16\pi G_N} \int d^4x\sqrt{-g}R, \tag{160}$$

where $g \equiv \det(g_{\mu\nu})$, G_N denotes Newton’s constant, and M_P is the reduced Planck mass (with $c = 1, \hbar = 1$),

其中 $g \equiv \det(g_{\mu\nu})$, G_N 表示牛顿引力常数，M_P 是约化普朗克质量 (满足 $c = 1, \hbar = 1$),

$$M_P = \frac{1}{\sqrt{8\pi G_N}} = 2.44 \cdot 10^{18}\mathrm{GeV}. \tag{161}$$

Vierbein and Cartan’s Formalism

标架与嘉当形式体系

In order to describe the coupling to fermions, an orthonormal basis of tangent vectors, e_a $(a = 0, 1, 2, 3)$, is introduced at each point of the spacetime manifold,

为了描述费米子的耦合，我们在时空流形的每一点引入一组切向量的标准正交基 e_a $(a = 0, 1, 2, 3)$,

$$g(e_a, e_b) = \eta_{ab}, \tag{162}$$

where

其中

$$\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1) \tag{163}$$

is the flat Minkowski metric. Orthonormality fixes the e_a only up to local Lorentz transformations, $e_a \to e_b \Lambda^b{}_a(x)$, with $\Lambda^b{}_a(x) \in SO(1, 3)$.

是平坦闵氏度规。标准正交性仅将 e_a 确定到差一个局域洛伦兹变换 $e_a \to e_b \Lambda^b{}_a(x)$ 的程度，满足 $\Lambda^b{}_a(x) \in SO(1, 3)$。

The basis change from e_a to the coordinate basis vectors ∂_μ and vice versa is written as

从 e_a 到坐标基矢 ∂_μ 的基变换 (反之亦然) 可写为

$$e_a = e_a^\mu(x)\,\partial_\mu, \;\; \partial_\mu = e_\mu^a(x)\,e_a, \tag{164}$$

where $e_\mu^a e_a^\nu = \delta_\mu^\nu, e_a^\mu e_\mu^b = \delta_a^b$. By a common abuse of terminology, we refer to both the vectors e_a and the matrix of conversion coefficients, $e_\mu^a(x)$, as the vierbein.

其中 $e_\mu^a e_a^\nu = \delta_\mu^\nu, e_a^\mu e_\mu^b = \delta_a^b$。按照术语的常用习惯性用法，我们将向量 e_a 和转换系数矩阵 $e_\mu^a(x)$ 都称为标架。

The vierbein e_μ^a and its inverse can be used to convert the ”curved” (or ”world”) indices $\mu, \nu, \ldots$ of any tensor field to ”flat” (or ”local Lorentz”) indices $a, b, \ldots$, in particular,

可以利用标架 e_μ^a 及其逆，将任意张量场的「弯曲」(或「世界」) 指标 $\mu, \upsilon, \ldots$ 转换为「平坦」(或「局域洛伦兹」) 指标 $a, b, \ldots$， 特别地,

$$g_{\mu\upsilon}(x) = e_\mu^a(x) e_\upsilon^b(x) \eta_{ab}, \tag{165}$$

which also implies

由此还可得

$$\sqrt{-g} \equiv \sqrt{-\det\left(g_{\mu\nu}\right)} = \det e_\mu^a \equiv e \tag{166}$$

and shows that the vierbein e_μ^a contains the same information as the metric.

这说明标架 e_μ^a 包含了和度规完全相同的信息。

When tensor fields are expressed in terms of the flat indices $a, b, \ldots$, the connection coefficients of the covariant derivative are denoted as ${\omega_\mu}^a{}_b$, so that, e.g., $\nabla_\mu V^a = \partial_\mu V^a + {\omega_\mu}^a{}_b V^b$. This covariant derivative is equivalent to the one for curved indices as defined by the Christoffel symbols (in the sense that $\nabla_\mu V^a = e_\upsilon^a \nabla_\mu V^\upsilon$, etc.) if

当张量场用平坦指标 $a, b, \ldots$ 表示时，协变导数的联络系数记为 ${\omega_\mu}^a{}_b$， 因此例如有 $\nabla_\mu V^a = \partial_\mu V^a + {\omega_\mu}^a{}_b V^b$。如果满足下述条件，该协变导数就等价于克里斯托费尔符号定义的弯曲指标协变导数 (在 $\nabla_\mu V^a = e_\upsilon^a \nabla_\mu V^\upsilon$ 等等成立的意义上):

$$\nabla_\mu e_\upsilon^a \equiv \partial_\mu e_\upsilon^a + {\omega_\mu}^a{}_b e_\upsilon^b - \Gamma_{\mu\upsilon}^\rho e_\rho^a = 0, \tag{167}$$

where, as suggested by the notation, the left-hand side can be viewed as a total covariant derivative that acts on curved and flat indices of the vierbein, implementing covariance with respect to general coordinate and local Lorentz transformations. Equation (167) is sometimes referred to as the ”vierbein postulate.”

其中，就像记号所提示的，左侧可看作作用在标架的弯曲指标与平坦指标上的全协变导数，实现了相对于广义坐标变换和局域洛伦兹变换的协变性。式 (167) 有时被称为「标架假设」。

In supergravity, it is useful to define also a derivative operator, D_μ, that is covariant only with respect to local Lorentz transformations, but not necessarily with respect to general coordinate transformations, i.e., all local Lorentz indices will be contracted with spin connections, but there are no Christoffel symbols that contract any world index. An important example is the Lorentz-covariant derivative of the vierbein e_υ^a,

在超引力中，定义另一个导数算符 D_μ 是很有用的，该算符仅对局域洛伦兹变换协变，不要求对广义坐标变换协变: 也就是说，所有局域洛伦兹指标都会和自旋联络缩并，但是没有克里斯托费尔符号缩并世界指标。一个重要例子是标架 e_υ^a 的洛伦兹协变导数,

$$D_\mu e_\upsilon^a \equiv \partial_\mu e_\upsilon^a + {\omega_\mu}^a{}_b e_\upsilon^b. \tag{168}$$

In view of (167), this expression does not vanish, but is equal to $\Gamma^{\rho}_{\mu\nu} e^{a}_{\rho}$. Recalling that the antisymmetrization $\Gamma^{\rho}_{[\mu\nu]} = \frac{1}{2} T_{\mu\nu}{}^{\rho}$ is just the torsion tensor of this connection, we can thus write

根据式 (167)，该表达式并不为零，而是等于 $\Gamma^{\rho}_{\mu\nu} e^{a}_{\rho}$ 。回忆到反对称化 $\Gamma^{\rho}_{[\mu\nu]} = \frac{1}{2} T_{\mu\nu}{}^{\rho}$ 恰好就是该联络的挠率张量，因此我们可以写出

$$D_{[\mu} e^{a}_{\nu]} = \frac{1}{2} T_{\mu\nu}{}^{a} = \frac{1}{2} T_{\mu\nu}{}^{\rho} e^{a}_{\rho}, \tag{169}$$

which is now a proper tensor field that vanishes for the Levi-Civita connection.

这是一个真张量场，对于列维-奇维塔联络该张量场为零。

Just as for the Levi-Civita connection in curved indices, the vanishing of the torsion tensor can be used to deduce an explicit expression for the spin connection in terms of the vierbein, $\omega_{\mu}{}^{ab} = \omega_{\mu}{}^{ab}(e)$. To this end, one considers

和弯曲指标下的列维-奇维塔联络一样，利用挠率张量为零的条件可以推导出自旋联络用标架 $\omega_{\mu}{}^{ab} = \omega_{\mu}{}^{ab}(e)$ 表示的显式表达式。为此，我们考虑

$$t^{dc,a} \equiv e^{\nu[d} e^{\mu c]} \left(\partial_{\mu} e^{a}_{\nu} + \omega_{\mu}{}^{a}{}_{b} e^{b}_{\nu} \right),$$

which is zero because of the assumed vanishing torsion. This then also implies the vanishing of the following combination:

由于假设挠率为零，该式等于零，由此也可以推出下述组合为零:

$$t^{dc,a} - t^{ca,d} - t^{ad,c} = 0, \tag{170}$$

in which only one term in the spin connection, $e^{\rho a} \omega_{\rho}{}^{cd}$, survives so that, after multiplying by $e_{\mu a}$, one obtains

其中自旋联络中只有 $e^{\rho a} \omega_{\rho}{}^{cd}$ 这一项保留下来，因此乘以 $e_{\mu a}$ 后可得

$$\omega_{\mu}{}^{cd}(e) = 2 e^{\nu[c} \partial_{[\mu} e^{d]}_{\nu]} - e_{a\mu} e^{\nu[c} e^{\sigma d]} \partial_{\nu} e^{a}_{\sigma}. \tag{171}$$

This connection is called the torsion-free spin connection, whose equivalence to the Levi-Civita connection (156) can also be verified directly.

该联络称为无挠自旋联络，可直接验证它等价于 (156) 的列维-奇维塔联络。

As mentioned above, even though the Lorentz-covariant derivative, D_{μ} , is not covariant with respect to general coordinate transformations, the antisymmetrized derivative (169) still forms a proper tensor field. In supergravity, all equations can similarly be expressed in terms of antisymmetrized Lorentz-covariant derivatives only so that the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ never occur. This allows one to use compact differential form notation, which can substantially reduce the index clutter in computations. To this end, we introduce the co-frame of one-forms, e^a , dual to the vectors e_b ,

上文提到，即使洛伦兹协变导数 D_μ 不满足一般坐标变换下的协变性，反对称化导数 (169) 仍是合格的张量场。类似地，超引力中所有方程都可以仅用反对称化洛伦兹协变导数表示，永远不会出现克里斯托费尔符号 $\Gamma^\rho_{\mu\nu}$ 。这允许我们使用紧凑的微分形式记号，大幅减少计算中的指标冗余。为此我们引入余标架的一元形式 e^a ， 它是向量 e_b 的对偶，

$$e^a \equiv e^a_\mu dx^\mu \tag{172}$$

as well as the connection one-forms,

以及联络一元形式:

$$\omega^a{}_b \equiv \omega_\mu{}^a{}_b dx^\mu. \tag{173}$$

In terms of e^a , the expression (169) for the torsion tensor reads

利用 e^a ， 挠张量的表达式 (169) 可写为

$$De^a \equiv de^a + \omega^a{}_b \wedge e^b = T^a = \frac{1}{2} dx^\mu \wedge dx^\upsilon T_{\mu\upsilon}{}^a, \tag{174}$$

whereas the curvature tensor of the connection $\omega_\mu{}^b{}_a$ is given by the two-form

而联络 $\omega_\mu{}^b{}_a$ 的曲率张量由二元形式给出

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{175}$$

The conditions for the connection $\omega_\mu{}^b{}_a$ to be equivalent to the Levi-Civita connection can now be written as

联络 $\omega_\mu{}^b{}_a$ 等价于列维-奇维塔联络的条件现在可写为

- $D\eta_{ab} \equiv d\eta_{ab} + \omega_a{}^c\eta_{cb} + \omega_b{}^c\eta_{ac} = 0$ (metric compatibility).

  - $D\eta_{ab} \equiv d\eta_{ab} + \omega_a{}^c\eta_{cb} + \omega_b{}^c\eta_{ac} = 0$ (度规相容性)。

- $T^a = 0$ (vanishing torsion).

  - $T^a = 0$ (挠为零)。

The first condition implies that the spin connection is antisymmetric in its indices, $\omega^{(ab)} = 0$, when both are raised or lowered with η , and the second equation (in combination with (174)) is equivalent to

第一个条件意味着自旋联络的指标满足反对称性: 当两个指标都用 η 升降时，有 $\omega^{(ab)}=0$；第二个条件结合式 (174) 等价于

$$De^a = 0\,. \tag{176}$$

In Cartan's formalism, the Einstein-Hilbert action reads

在嘉当框架下，爱因斯坦-希尔伯特作用量可写为

$$S_{EH} = \frac{M_P^2}{4}\int R^{ab}\wedge e^c\wedge e^d\varepsilon_{abcd}, \tag{177}$$

where $\varepsilon_{0123}=1$, and the standard form (160) is recovered via the identification of the four-dimensional measure,

其中 $\varepsilon_{0123}=1$ ，通过识别四维测度可以得到标准形式 (160):

$$dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma = -d^4xe\varepsilon^{\mu\nu\rho\sigma}, \tag{178}$$

with $\varepsilon_{\mu\upsilon\rho\sigma} = e^a_\mu e^b_\upsilon e^c_\rho e^d_\sigma\varepsilon_{abcd}$.

其中 $\varepsilon_{\mu\upsilon\rho\sigma} = e^a_\mu e^b_\upsilon e^c_\rho e^d_\sigma\varepsilon_{abcd}$ 。

Considering the vierbein and spin connection as independent fields yields the field equations

将标架和自旋联络视为独立场可得到场方程

$$\frac{\delta S}{\delta e^a_\mu} = 0 \Leftrightarrow G_{\mu\nu} = 0, \tag{179}$$

$$\frac{\delta S}{\delta\omega^{ab}_\mu} = 0 \Leftrightarrow T^a = 0, \tag{180}$$

where

其中

$$G_{\mu\upsilon} \equiv R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R \tag{181}$$

is the usual Einstein tensor. Hence, in this formalism, ${\omega_\mu}^{ab}$ is dynamically fixed to be the torsion-free spin connection, and one recovers the standard Einstein equations.

是通常的爱因斯坦张量。因此在该框架下，${\omega_\mu}^{ab}$ 动力学固定为无挠自旋联络，我们可以重新得到标准爱因斯坦方程。

In supergravity, on the other hand, the spin connection also appears in the kinetic term of the gravitino so that Eq. (180) receives a correction term bilinear in the gravitino fields. Thus, in supergravity, the spin connection as determined from the field equation has torsion and is no longer equivalent to the torsion-free Levi-Civita connection. This is not a problem, because the Christoffel symbols never appear in the theory, and one is always free to redefine the spin connection as

另一方面，在超引力中，自旋联络也出现在引力微子的动能项中，因此式 (180) 会得到一个引力微子场的双线性修正项。因此，在超引力中，由场方程确定的自旋联络具有挠率，不再等价于无挠的列维-奇维塔联络。这并不是一个问题，因为该理论中从不出现克里斯托费尔符号，我们总可以自由地将自旋联络重新定义为

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + \kappa_\mu{}^{ab}, \tag{182}$$

where $\omega_\mu{}^{ab}(e)$ is the torsion-free spin connection (171) and $\kappa_\mu{}^{ab}$ is the contorsion tensor,

其中 $\omega_\mu{}^{ab}(e)$ 是无挠自旋联络 (171)，$\kappa_\mu{}^{ab}$ 是逆挠张量,

$$\kappa_\mu{}^{ab} = e_\mu^c \left(T^{ab}{}_c - T_c{}^{ab} - T^b{}_c{}^a\right), \tag{183}$$

bilinear in the fermions. Expressing the theory in terms of $\omega_\mu{}^{ab}(e)$ then introduces explicit four-Fermion interactions in the action.

它是费米子的双线性项。将理论用 $\omega_\mu{}^{ab}(e)$ 表示后，就会在作用量中引入显式的四费米子相互作用。

Gamma Matrices

伽马矩阵

We first introduce gamma matrices in flat Minkowski space with orthonormal basis vectors labelled by the indices $(a, b, \ldots = 0, 1, 2, 3)$ and the standard Minkowski metric

我们首先在平坦闵可夫斯基空间中引入伽马矩阵, 该空间的标准正交基矢由指标 $(a, b, \ldots = 0, 1, 2, 3)$ 标记，且采用标准闵可夫斯基度规

$$\eta_{ab} = \mathrm{diag}\left(-1, +1, +1, +1\right). \tag{184}$$

The Lorentz algebra generators are denoted by $M_{ab} = -M_{ba}$ and satisfy

洛伦兹代数生成元记为 $M_{ab} = -M_{ba}$ ，满足

$$[M_{ab}, M_{cd}] = -2\eta_{c[a}M_{b]d} + 2\eta_{d[a}M_{b]c}. \tag{185}$$

The gamma matrices, γ_a , obey following basic relations:

伽马矩阵 γ_a 遵循以下基本关系:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1}_4, \tag{186}$$

$$\gamma_0^\dagger = -\gamma_0,\ \gamma_i^\dagger = +\gamma_i,\ \gamma_a^T = \pm\gamma_a, \tag{187}$$

$$\sum\nolimits_{ab} := \frac{1}{4}[\gamma_a, \gamma_b] \tag{188}$$

$$\gamma_{a_1\ldots a_p} \equiv \gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_p]}, \tag{189}$$

$$\gamma_5 \equiv \gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = +i\gamma_0\gamma_1\gamma_2\gamma_3, \tag{190}$$

$$(\gamma_5)^2 = \mathbb{1}_4,\ \{\gamma_5, \gamma_a\} = 0,\ \left[\gamma_5, \sum\nolimits_{ab}\right] = 0. \tag{191}$$

We also introduce the charge conjugation matrix C , satisfying

我们还引入电荷共轭矩阵 C ，其满足

$$C^T = -C = C^{-1} = C^\dagger, \tag{192}$$

$$\gamma_a^T = -C\gamma_a C^{-1} \tag{193}$$

implying the following symmetry properties:

由此可得下列对称性:

$$C^T = -C,\ (C\gamma^a)^T = (C\gamma^a),\ \left(C\gamma^{ab}\right)^T = \left(C\gamma^{ab}\right),$$

(194)

$$\left(C\gamma^{abc}\right)^T = -\left(C\gamma^{abc}\right),\ \left(C\gamma^{abcd}\right)^T = -\left(C\gamma^{abcd}\right).$$

The matrix γ_5 also enters a number of useful duality relations between the antisymmetrized products of gamma matrices,

矩阵 γ_5 还出现在伽马矩阵反对称乘积之间的诸多有用对偶关系中,

$$\gamma^{abc} = i\varepsilon^{abcd}\gamma_d\gamma_5,\ i\gamma_a\gamma_5 = \frac{1}{3!}\varepsilon_{abcd}\gamma^{bcd},$$

$$\gamma^{abcd} = -i\varepsilon^{abcd}\gamma_5,\ i\gamma_5 = \frac{1}{4!}\varepsilon_{abcd}\gamma^{abcd}, \tag{195}$$

$$\gamma^{ab} = \frac{i}{2}\varepsilon^{abcd}\gamma_{cd}\gamma_5$$

where ε_{abcd} is the antisymmetric epsilon tensor with

其中 ε_{abcd} 是反对称 epsilon 张量，满足

$$\varepsilon_{0123} = 1\,. \tag{196}$$

Spinors

旋量

Throughout this text, we use the four-component spinor notation. A translation to the two-component spinor notation in our conventions can be found in [1].

全文采用四分量旋量记号，在本文约定下转换为二分量旋量记号的相关内容可见文献 [1]。

Dirac Spinors

狄拉克旋量

Dirac spinors are elements of the representation space of the Clifford algebra (186). As the matrices Σ_{ab} provide an explicit linear representation of the Lorentz algebra generators M_{ab} , Dirac spinors inherit also a Lorentz algebra representation.

狄拉克旋量是克利福德代数表示空间的元素 (186)。由于矩阵 Σ_{ab} 给出了洛伦兹代数生成元 M_{ab} 的一个显式线性表示，狄拉克旋量也继承了洛伦兹代数表示。

For a generic Dirac spinor, ψ , we define the Dirac conjugate as

对于任意狄拉克旋量 ψ ，我们将狄拉克共轭定义为

$$\overline{\psi} \equiv i\psi^\dagger\gamma^0 = -i\psi^\dagger\gamma_0 \tag{197}$$

so that bilinears such as $\overline{\psi}\chi$ are Lorentz-invariant, because of $\Sigma_{ab}^\dagger\gamma_0 = -\gamma_0\Sigma_{ab}$.

使得形如 $\overline{\psi}\chi$ 的双线性场是洛伦兹不变量，这由 $\Sigma_{ab}^\dagger\gamma_0 = -\gamma_0\Sigma_{ab}$ 保证。

On 4D Dirac spinors, however, the Lorentz algebra representation provided by Σ_{ab} is reducible. This reducibility can be resolved either by a chirality condition, leading to Weyl spinors, or by a reality condition, leading to Majorana spinors.

然而在四维狄拉克旋量上，Σ_{ab} 给出的洛伦兹代数表示是可约的。这种可约性可以通过两种方式处理: 手征条件得到外尔旋量，或是实条件得到马约拉纳旋量。

The Weyl Condition

外尔条件

The Weyl condition projects out the part of a spinor that has a particular handedness. In order to impose it, one uses the chirality projectors

外尔条件可以投影出旋量中特定手征的分量。我们可以利用手征投影算子来施加该条件

$$P_L \equiv \frac{1}{2}\left(1+\gamma^5\right), \; P_R \equiv \frac{1}{2}\left(1-\gamma^5\right) \tag{198}$$

and defines left- and right-handed spinors,

并定义左手旋量与右手旋量，

$$\psi_L \equiv P_L\psi, \; \psi_R \equiv P_R\psi, \; \text{(Weyl condition)} \tag{199}$$

satisfying $\gamma^5\psi_L = \psi_L$ and $\gamma^5\psi_R = -\psi_R$. This projection is consistent with Lorentz covariance, and left- and right-handed spinors form separate representations of the Lorentz group.

满足 $\gamma^5\psi_L = \psi_L$ 和 $\gamma^5\psi_R = -\psi_R$ 。该投影满足洛伦兹协变性，左手旋量与右手旋量分别构成洛伦兹群的独立表示。

Using $\gamma_5^\dagger = \gamma_5$ as well as $P_L\gamma_0 = \gamma_0 P_R$ and $P_R\gamma_0 = \gamma_0 P_L$, one finds

利用 $\gamma_5^\dagger = \gamma_5$ 以及 $P_L\gamma_0 = \gamma_0 P_R$ 和 $P_R\gamma_0 = \gamma_0 P_L$ ，可得

$$\overline{\psi_R} = \overline{\psi} P_L, \; \overline{\psi_L} = \overline{\psi} P_R, \tag{200}$$

where

其中

$$\overline{\psi_R} \equiv \overline{(P_R\psi)} = -i(P_R\psi)^\dagger\gamma_0. \tag{201}$$

Because of this, we will often write

因此，我们通常会写成

$$\overline{\psi}_L \equiv \overline{\psi} P_L = \overline{\psi_R}, \; \overline{\psi}_R \equiv \overline{\psi} P_R = \overline{\psi_L}. \tag{202}$$

The Majorana Condition

马约拉纳条件

The Majorana condition is a reality condition that is usually expressed in terms of the charge conjugation matrix, C . In terms of C , the charge conjugate spinor of a four-component spinor, ψ , is defined as

马约拉纳条件是一种实条件，通常通过电荷共轭矩阵 C 表述。以 C 为基础，四分量旋量 ψ 的电荷共轭旋量定义为

$$\psi^c = C\overline{\psi}^T = iC\gamma^{0T}\psi^*. \tag{203}$$

A Majorana spinor is then a spinor that equals its own charge conjugate,

马约拉纳旋量即是等于自身电荷共轭的旋量，

$$\psi^c = \psi. \text{ (Majorana condition)} \tag{204}$$

For a Majorana spinor, the Dirac conjugate (197) takes on the simple form

对于马约拉纳旋量，狄拉克共轭 (197) 可以简化为

$$\overline{\psi} = \psi^T C. \tag{205}$$

For anti-commuting Majorana spinors, this then implies

对于反对易马约拉纳旋量，由此可以得到

$$\overline{\psi}_1 M \psi_2 = \begin{cases} +\overline{\psi}_2 M \psi_1 & \text{for } M = \mathbb{1}_4, \gamma_{abc}, \gamma_{al} \\ -\overline{\psi}_2 M \psi_1 & \text{for } M = \gamma_a, \gamma_{ab} \end{cases} \tag{206}$$

Unless stated otherwise, we will always use anti-commuting Majorana spinors, but often also take in addition the chiral projections ψ_L and ψ_R of these Majorana spinors, which therefore are not independent.

除非另有说明，我们始终使用反对易马约拉纳旋量，同时也常会额外对这些马约拉纳旋量做手征投影 ψ_L 和 ψ_R ，因此这两个投影不是独立的。

More specifically, the chiral projections, ψ_L and ψ_R , of a Majorana spinor ψ satisfy

更具体地说，马约拉纳旋量 ψ 的手征投影 ψ_L 和 ψ_R 满足

$$(\psi_L)^c = \psi_R, \ (\psi_R)^c = \psi_L, \tag{207}$$

showing that ψ_L and ψ_R are no longer Majorana spinors themselves. In other words, a spinor in 4D cannot be simultaneously chiral and Majorana, but it nevertheless does make sense to talk about the chiral projections ψ_L or ψ_R of a given Majorana spinor ψ .

这说明 ψ_L 和 ψ_R 本身不再是马约拉纳旋量。换句话说，四维中的旋量不可能同时是手征旋量和马约拉纳旋量，但讨论给定马约拉纳旋量 ψ 的手征投影 ψ_L 或 ψ_R 仍然是有意义的。

In fact, several identities that hold for Majorana spinors still have a close analogue for their chiral projections. In particular, the chiral projection ψ_L of a Majorana spinor ψ satisfies

事实上，马约拉纳旋量满足的许多恒等式，对其手征投影依然有相近的对应形式。尤其是，马约拉纳旋量 ψ 的手征投影 ψ_L 满足

$$\overline{\psi}_L \equiv \overline{\psi} P_L = (\psi_L)^T C \tag{208}$$

and similarly for ψ_R , as follows from (205) and $C\gamma_5 = \gamma_5^T C$. From this, one can obtain more symmetry properties for the chiral projections that are very similar to Eqs. (206) for the Majorana spinors themselves,

ψ_R 也有类似的结论，可由 (205) 和 $C\gamma_5 = \gamma_5^T C$ 推得。由此我们可以得到手征投影更多的对称性，这些性质和马约拉纳旋量本身对应的式 (206) 非常相似，

$$\begin{gathered}
\overline{\chi}_L \psi_L = \chi_L^T C \psi_L = -\psi_L^T C^T \chi_L = \overline{\psi}_L \chi_L \\
\overline{\chi}_L \gamma^a \psi_R = -\overline{\psi}_R \gamma^a \chi_L,\ \overline{\chi}_L \gamma^{ab} \psi_L = -\overline{\psi}_L \gamma^{ab} \chi_L \\
\overline{\chi}_L \gamma^{abc} \psi_R = \overline{\psi}_R \gamma^{abc} \chi_L.
\end{gathered} \tag{209}$$

Spinors in Curved Spacetime

弯曲时空中的旋量

Spinors are double-valued representations of the Lorentz group. On a curved manifold, the relevant Lorentz transformations are the local Lorentz transformations of the vierbein so that

旋量是洛伦兹群的双值表示。在弯曲流形上，相关的洛伦兹变换是四标架的局域洛伦兹变换，因此

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \psi \tag{210}$$

is the proper Lorentz-covariant derivative of a fermion field ψ .

是费米子场 ψ 的适当洛伦兹协变导数。

The gamma matrices with a curved index μ are obtained from the constant γ_a via contaction with a vierbein:

带弯曲指标 μ 的伽马矩阵可通过常数 γ_a 与四标架缩并得到:

$$\gamma_\mu \equiv e_\mu^a \gamma_a \tag{211}$$

The γ_μ are then in general no longer constant, and they transform non-trivially under a variation of the vierbein.

因此 γ_μ 一般不再是常数，它在四标架变换下会发生非平凡变换。

Fierz Identities

菲尔茨恒等式

We will often need to rewrite three- or four-fermion terms to complete our analysis of the supersymmetry properties of an action, and hence, Fierz identities will be extremely useful. We list here the main ones for two spinors:

在分析作用量的超对称性质时，我们经常需要改写三费米子项或四费米子项，因此菲尔茨恒等式非常有用。我们在此列出针对两个旋量的主要恒等式:

$$\psi_R \overline{\chi}_R = -\frac{1}{2}\overline{\chi}_R \psi_R P_R + \frac{1}{8}\overline{\chi}_R \gamma_{ab} \psi_R \gamma^{ab} P_R, \tag{212}$$

$$\psi_R \overline{\chi}_L = -\frac{1}{2}\overline{\chi}_L \gamma^a \psi_R \gamma_a P_L, \tag{213}$$

where for the sake of clarity, we explicitly left the projectors on the right-hand side.

为清晰起见，我们在右侧显式保留了投影算子。

We often make use of spinor one-forms $\psi = dx^\mu \psi_\mu$. Exchanging such spinor one-forms then leads to an additional minus sign from the anti-commutativity of the wedge product, and hence, one has

我们经常使用旋量一形式 $\psi = dx^\mu \psi_\mu$ 。交换这类旋量一形式时，楔积的反对易性会带来一个额外的负号，因此有

$$\psi_R \wedge \overline{\psi}_R = -\frac{1}{8}\overline{\psi}_R \wedge \gamma_{ab} \psi_R \gamma^{ab} P_R, \tag{214}$$

$$\psi_R \wedge \overline{\psi}_L = \frac{1}{2}\overline{\psi}_L \wedge \gamma^a \psi_R \gamma_a P_L \tag{215}$$

where now $\overline{\psi}_L \wedge \psi_L = 0$ because of (209) and the wedge product. A crucial consequence is the so-called cyclic identity:

结合式 (209) 与楔积的性质，此处得到 $\overline{\psi}_L \wedge \psi_L = 0$ 。一个重要推论是所谓的循环恒等式:

$$\gamma^a \psi_L \wedge \overline{\psi}_L \wedge \gamma_a \psi_R = 0. \tag{216}$$

Lie Derivative on P-Forms

p 形式的李导数

The Lie derivative of a p-form A_p along the flow of a vector field V is defined as

p 形式 A_p 沿向量场 V 流的李导数定义为

$$L_V A_p = \lim_{t \to 0} \frac{1}{t} \left(\sigma_t^* A_p \left(\sigma_t \left(x \right) \right) - A_p \left(x \right) \right),$$

where σ_t^* is the pull back of the differential form along the flow generated by the vector field V . When applied to a scalar valued p -form, this reduces to

其中 σ_t^* 是微分形式沿向量场 V 生成流的拉回。将其作用于标量值 p 形式时，可化简为

$$L_V A_p = (\iota_V d + d\iota_V) A_p.$$

The Supersymmetry Algebra

超对称代数

The $\mathcal{N} = 1$ supersymmetry algebra in four dimensions can be represented as follows:

四维下的 $\mathcal{N} = 1$ 超对称代数可以表示为如下形式:

$$\{Q, \bar{Q}\} = -2i\gamma^a \mathcal{P}_a$$

$$[\mathcal{P}_a, Q] = 0$$

$$[\mathcal{M}_{ab}, Q] = \frac{i}{2} \gamma_{ab} Q$$

$$[R, Q] = i\gamma_5 Q, \tag{217}$$

$$[\mathcal{P}_a, \mathcal{P}_b] = 0$$

$$[\mathcal{P}_a, \mathcal{M}_{bc}] = -2i\eta_{a[b}\mathcal{P}_{c]}$$

$$[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = 2i\eta_{c[a}\mathcal{M}_{b]d} - 2i\eta_{d[a}\mathcal{M}_{b]c},$$

where we used Hermitian generators $\mathcal{P}_a$ and $\mathcal{M}_{ab}$ for the Poincaré algebra. Just as in (185), we sometimes also use their anti-Hermitian counterparts,

其中我们对庞加莱代数采用了厄米生成元 $\mathcal{P}_a$ 和 $\mathcal{M}_{ab}$ 。就像式 (185) 中那样，我们有时也会使用它们的反厄米形式,

$$P_a = i\mathcal{P}_a,\ M_{ab} = i\mathcal{M}_{ab} \tag{218}$$

when this is more convenient.

当这样处理更为方便时。

Cross-References

交叉引用

Covariant Superspace Approaches to $\mathcal{N} = 2$ Supergravity

$\mathcal{N} = 2$ 超引力的协变超空间方法

Croup Manifold Approach to Supergravity

超引力的群流形方法

Inflationary Cosmology from Supergravity

源自超引力的暴胀宇宙学

- $\mathcal{N} = 2$ Supergravities in Harmonic Superspace

  - 调和超空间中的 $\mathcal{N} = 2$ 超引力

11D Supergravity and Hidden Symmetries

11 维超引力与隐藏对称性

The AdS/CFT Correspondence

AdS/CFT 对应

Acknowledgments This review is based on the book [1], which grew out of the many interactions we had in our careers. We would only like here to recall those who played a pivotal role in shaping our understanding of supergravity: A. Ceresole, R. D'Auria, S. Ferrara, M. Günaydin, A. Van Proeyen, and F. Zwirner. We also thank once more L. Álvarez Gaumé, who first suggested to turn our lecture notes into a book.

致谢这篇综述基于著作 [1]，它诞生于我们职业生涯中的诸多交流探讨。在此我们仅想要提及那些对塑造我们对超引力的理解起到关键作用的学者:A. Ceresole、R. D'Auria、S. Ferrara、M. Günaydin、A. Van Proeyen 和 F. Zwirner。我们也要再次感谢 L. Álvarez Gaumé，是他最先提议我们将讲义整理成书。

References

参考文献

1. G. Dall'Agata, M. Zagermann, Supergravity: from first principles to modern applications. Lect. Notes Phys. 991, 1-263 (2021). ISBN 978-3-662-63978-8, 978-3-662-63980-1

2. D.Z. Freedman, P. van Nieuwenhuizen, S. Ferrara, Progress toward a theory of supergravity. Phys. Rev. D13, 3214-3218 (1976)

3. S. Deser, B. Zumino, Consistent supergravity. Phys. Lett. B62, 335 (1976)

4. P. Van Nieuwenhuizen, Supergravity. Phys. Rept. 68, 189-398 (1981)

5. H.P. Nilles, Supersymmetry, supergravity and particle physics. Phys. Rept. 110, 1 (1984)

6. B. de Wit, D.Z. Freedman, Supergravity: The basics and beyond, Bonn Superym.ASI 1984:0135. MIT-CTP-1238

7. L. Castellani, R. D'Auria, P. Fre, Supergravity and Superstrings: A Geometric Perspective. Mathematical Foundations, vol. 1 (World Scientific, Singapore, 1991), pp. 1-603

8. L. Castellani, R. D'Auria, P. Fre, Supergravity and Superstrings: A Geometric Perspective. Supergravity, vol. 2 (World Scientific, Singapore, 1991), pp. 607-1371

9. J. Wess, J. Bagger, Supersymmetry and Supergravity (Princeton University Press, Princeton, New Jersey, 1992)

10. A. Van Proeyen, Tools for Supersymmetry. hep-th/9910030

11. J.-P. Derendinger, Introduction to Supergravity, Lectures at the Summer School Gif 2000 in Paris

12. B. de Wit, Supergravity. hep-th/0212245

13. A. Van Proeyen, Structure of Supergravity Theories. To be published in the series Publications of the Royal Spanish Mathematical Society. hep-th/0301005

14. W. Rarita, J. Schwinger, On a theory of particles with half integral spin. Phys. Rev. 60, 61 (1941)

15. M. Fierz, On the relativistic theory of force-free particles with any spin. Helvetica Phys. Acta 12(I), 3-37 (1939); M. Fierz, W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. Proc. R. Soc. Lond. A 173, 211 (1939)

16. M.A. Vasiliev, Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions. Int. J. Mod. Phys. D 5, 763-797 (1996). [arXiv:hep-th/9611024 [hep-th]]; Higher spin gauge theories: star product and AdS space. [arXiv:hep-th/9910096 [hep-th]]

17. A.H. Chamseddine, P.C. West, Nucl. Phys. B 129, 39 (1977)

18. S.W. MacDowell, F. Mansouri, Phys. Rev. Lett. 38, 739 (1977); [Erratum: Phys. Rev. Lett. 38, 1376 (1977)]

19. T.W.B. Kibble, J. Math. Phys. 2, 212-221 (1961). https://doi.org/10.1063/1.1703702

20. P. van Nieuwenhuizen, Supergravity as a Yang-Mills Theory. hep-th/0408137

21. S. Ferrara, Algebraic properties of extended supergravity in de sitter space. Phys. Lett. B69, 481 (1977)

22. K. Pilch, P. van Nieuwenhuizen, M.F. Sohnius, De sitter superalgebras and supergravity. Commun. Math. Phys. 98, 105 (1985)

23. I. Antoniadis, E. Dudas, S. Ferrara, A. Sagnotti, The Volkov-Akulov-Starobinsky supergravity. Phys. Lett. B 733, 32-35 (2014). [arXiv:1403.3269 [hep-th]]

24. E.A. Bergshoeff, D.Z. Freedman, R. Kallosh, A. Van Proeyen, Pure de sitter supergrav-ity. Phys. Rev. D 92(8), 085040 (2015); [erratum: Phys. Rev. D 93(6), 069901 (2016)]. [arXiv:1507.08264 [hep-th]]

25. G. Dall'Agata, E. Dudas, F. Farakos, On the origin of constrained superfields. JHEP 05, 041 (2016). [arXiv:1603.03416 [hep-th]]

26. N. Cribiori, G. Dall'Agata, F. Farakos, From linear to non-linear SUSY and back again. JHEP 08, 117 (2017). [arXiv:1704.07387 [hep-th]]

27. P.K. Townsend, Cosmological constant in supergravity. Phys. Rev. D15, 2802-2804 (1977)

28. S. Cecotti, L. Girardello, M. Porrati, Constraints on partial superhiggs. Nucl. Phys. B268, 295-316 (1986)

29. E. Cremmer, S. Ferrara, L. Girardello, A. Van Proeyen, Coupling supersymmetric Yang-Mills theories to supergravity. Phys. Lett. B 116, 231-237 (1982)

30. E. Cremmer, S. Ferrara, L. Girardello, A. Van Proeyen, Yang-Mills theories with local supersymmetry: Lagrangian, transformation laws and superhiggs effect. Nucl. Phys. B 212, 413 (1983)

31. J.A. Bagger, Coupling the gauge invariant supersymmetric nonlinear sigma model to super-gravity. Nucl. Phys. B 211, 302 (1983)

32. H. Elvang, D.Z. Freedman, B. Kors, Anomaly cancellation in supergravity with Fayet-Iliopoulos couplings. JHEP 11, 068 (2006). [arXiv:hep-th/0606012 [hep-th]]

33. J. De Rydt, J. Rosseel, T.T. Schmidt, A. Van Proeyen, M. Zagermann, Symplectic structure of $N = 1$ supergravity with anomalies and Chern-Simons terms. Class. Quant. Grav. 24, 5201-5220 (2007). [0705.4216]

34. P. Binetruy, G. Dvali, R. Kallosh, A. Van Proeyen, Fayet-Iliopoulos terms in supergravity and cosmology. Class. Quant. Grav. 21, 3137-3170 (2004). [hep-th/0402046]